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Alfredo Salgado
Banco de México

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Incomplete Information and Costly Signaling in College Admissions*

Alfredo Salgado†
Banco de México

Abstract: We analyze a college admissions game with asymmetric information between students and colleges. Students' preferences for colleges depend on the observable quality of the schools. In contrast, colleges' preferences for students depend on the latter's abilities, which are private information. Students and schools are matched via a decentralized mechanism in which students signal their abilities with costly observable signals. A closed-form symmetric separating equilibrium of this game that depends on the supply of and demand for schools seats and on college quality is characterized. In this equilibrium, an increase in the number of students, a reduction in the number of school seats or a drop in the quality of schools reduce the incentive of low-ability students to invest in signaling and increase it for high-ability students.
Keywords: College Admissions, Decentralized Mechanisms, Incomplete Information, Coordination Problems, Costly Signaling
JEL Classification: D82, C70, C71, C72, C78

Resumen: Se analiza un juego de admisión a la universidad entre estudiantes y escuelas con información asimétrica. Las preferencias de los estudiantes por las universidades dependen de la calidad observable de las escuelas. En contraste, las preferencias de las universidades por los estudiantes dependen de las habilidades de estos últimos, que son información privada. Estudiantes y escuelas son emparejados mediante un mecanismo descentralizado en el que los estudiantes invierten en una señal observable y costosa de sus habilidades. Se analiza un equilibrio separador simétrico que depende de la oferta y la demanda por plazas escolares, así como de la calidad de las universidades. En este equilibrio, un aumento del número de estudiantes, una reducción de las vacantes o una caída en la calidad de las escuelas disminuyen el incentivo de los estudiantes con menores habilidades de invertir en señalización, mientras que lo aumentan para los de mayores habilidades.
Palabras Clave: Admisión a la Universidad, Mecanismos Descentralizados, Información Incompleta, Problemas de Coordinación, Señalización Costosa

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† Dirección General de Investigación Económica, Banco de México. Email: asalgadot@banxico.org.mx.

1 Introduction

In this paper, we analyze a decentralized two-sided matching problem in the context of college admissions (Gale and Shapley, 1962). Unlike previous literature, we consider the presence of incomplete information and the role of costly signaling to understand how colleges and students match with each other in this environment. We consider a setting where students want to enroll in colleges with observable quality while colleges seek to accept high-skilled students. We assume that students have private information about their academic skills. In addition, agents are matched according to a simple decentralized matching mechanism called Costly Signaling Mechanism (CSM) that runs in two stages. In the first signaling stage, students choose a costly observable score to signal their abilities. In the second matching stage, as in Alcalde and Romero-Medina (2000), colleges and students are matched according to a simple two-stage matching process. First, colleges simultaneously make an offer to a student, then students collect their offers and simultaneously choose one among the available ones. The CSM induces an extensive form game that is characterized by an equilibrium matching and a signaling strategy.

A decentralized two-sided matching problem can be understood as a way of organizing a match by letting agents on one side of the market approaching agents on the opposite side and propose a match. Several real-world decentralized matching problems that function in the same way can be observed. These include examples in the job market, marriage matching problems and college admissions, among other examples. Since agents are free to propose a match to anyone on the opposite side of the market and those who received a proposal are free to accept or reject it, a coordination problem may arise. As a consequence, some agents, that in general should be matched, might be left unpaired and the mechanism might be ineffective in matching agents efficiently according to a particular criteria. In addition, the matching mechanism can also match agents in an unstable way. The stability of the matching is a natural and desirable property of centralized and decentralized mechanisms (Alcade, 1996). A stable match is one where no matched agent prefers to stay alone and no pair of agents on opposite sides of the market prefer to be matched to each other rather than to their current pairs. Intuitively, whenever a match is stable, individual self-interest prevents any individual deviation or side deal that could break the current match.[1]

Several papers have analyzed the possibility of solving the issues associated with the presence of coordination problems and attaining stable assignments in decentralized matching markets. This literature shows that some simple matching mechanisms can solve these problems in environments with complete information. For instance, Alcalde, Pérez-Castrillo and Romero-Medina (1998) and Alcalde and Romero-Medina (2000) propose two very similar matching mechanisms in two stages that induce a game in extensive form. They show that these procedures implement the set of stable matchings in subgame perfect equilibrium. These matching mechanisms assure the stability of the equilibrium assignments by restricting agents to send only one application. Whenever agents can send multiple applications, unstable assignments may arise in equilibrium. Fortunately, according to Triossi (2009), it is easy to restore the stability of equilibrium assignments in this kind of decentralized mechanisms by introducing a small application cost. In addition, even when the application cost is negligible and students can submit multiple applications, some dynamic mechanisms are effective to reach stable equilibrium matches.[2]

According to the previous argument, it seems that in environments with complete information it is possible to find decentralized matching mechanisms that guarantee the stability of the equilibrium assignments and solve the coordination problem. However, in the presence of incomplete information, we require additional conditions to deal with these problems. Coles, Kushnir and Niederle (2010) introduce a cost-free signaling mechanism in decentralized matching problems with incomplete information about agents' preferences.[3] Among other desirable properties, this mechanism increases the expected number of matches and the welfare of agents who signal their preferences in equilibrium. However, a costly signaling

[1]The most important applications of the theory of stable matchings in real-life problems include the National Resident Matching Program (NRMP) and the NRMP with couples; the New York City public school system assignment; Boston's public school system assignment; the New England Program for Kidney Exchange, among other applications (Roth,2008).

[2]See Sotomayor, 2003; Romero-Medina and Triossi, 2014; and Haeringer, G. and Wooders, M., 2011.

[3]Another example can be found in Coles, et. al. (2010). They analyze the role of signaling preferences in the context of the Job Market for New Economist.

setting seems to be a more appropriate approach to analyze several real-life decentralized matching processes such as the college admissions problem, in the presence of private information. It is well known that most selective colleges and universities in the USA require a set of signals for college admissions that seem to be significantly costly for students. For instance, test scores of either the SAT or ACT,[4] essay questions, recommendation letters, and personal interviews, among other requirements (Dearden, Meyerhoefer and Yang, 2017). Hence, students have to spend a significant amount of effort (and money) to each application in order to signal their skills and improve their chance of being accepted at desirable colleges.[5]

In order to simplify the analysis, we consider a simple setting of a one-to-one matching problem, i.e. each student enrolls in at most one college and each college has only one available school seat. This assumption can be relaxed, as in previous literature, by assuming that colleges have responsive preferences over sets of students (Roth and Sotomayor, 1990). However, we consider that extending the model in this direction could make the characterization of the equilibrium much more complex and less intuitive without adding relevant insights for understanding the consequences of including incomplete information and costly signaling on decentralized college admissions. We also assume that all agents on each side of the market have the same preferences, i.e. all students rank colleges according to observable quality and every college would have the same ranking over students according to their academic skills if these were observable. While it seems quite reasonable to assume that colleges prefer high-skilled students over less skilled ones, assuming that all students have the same preferences over college quality is a strong assumption. However, there is empirical evidence in favor of this assumption. For instance, Griffith and Rask (2007) show that in the USA full-paying applicants are more likely to attend a school that has an even slightly higher rank (according to the U.S. News Ranking). Moreover, while applicants who received student aid are less responsive, they still systematically prefer higher ranked schools.

[4]The SAT (Scholastic Assessment Test) and the ACT (American College Testing) are the most important standardized tests for college admissions in the USA.

[5]See Bettinger and Baker (2011), Dominguez and Briggs (2009), Montgomery and Lilly (2012) and Powers and Rock (1999).

As a first approach to understand the effects of the presence of incomplete information in college admissions, we analyze a benchmark matching problem under the CSM with no signaling. In this setting, all students are identical ex-ante, since colleges only know the prior distribution of student abilities. Under these conditions, we characterize a symmetric equilibrium of this game in which agents' payoffs depend on the number of students, the expected value of students' skills and colleges' quality. This equilibrium has several interesting implications. First, in equilibrium colleges expect to enroll average-skilled students, since the matching process does not provide any additional information about students' abilities. Second, we find that the probability of enrolling a student is decreasing in college quality. Thus, only the highest quality college fills its seat with probability one while the rest of the agents may end up unmatched with positive probability. Finally, we show that an increase in the number of students increases colleges' payoffs but decreases those of students.

After analyzing the CSM with no signaling, we characterize a separating symmetric equilibrium of the game induced by the CSM where all students play according to the same signaling strategy. To sustain this separating equilibrium, we consider a set of beliefs where higher-scored students are associated with higher abilities. Under these beliefs, colleges form an interim ordinal preference relation on the set of students according to which they prefer to enroll higher-scored students. This implies that for each profile of student scores, there is a unique equilibrium assignment in the matching stage of the CSM that is consistent with those beliefs. This equilibrium assignment is assortative, i.e. the highest-scored student is matched with the best college, the second highest-scored student is matched with the second best college, and so on.[6]

In the signaling stage of the CSM, students take as given the assortative assignment of the matching stage and play a signaling game where they choose a costly observable score to signal their abilities.[7] We characterize a symmetric pure strategy Nash equilibrium of this game. This equilibrium is characterized by a strictly increasing, continuous, and differen-

[6]Hoppe, Moldovanu and Sela (2009) analyze a similar model where the assortative matching is exogenously given.

[7]This approach is similar to all-pay actions and contests models. See for instance: Barut and Kovenock (1998), Glazer and Hassin (1988) and Moldovanu and Sela (2001).

tiable signaling strategy that depends on student's skills. In equilibrium, no two students choose the same score, so that a unique equilibrium matching is induced that is assortative with respect to the true student's skills. That is, the highest-skilled student is matched with the best college, the second-highest skilled student is matched with the second best college, and so on. Further, in this equilibrium the number of potential matches is maximized and the outcome is stable.

Our closed-form characterization of the signaling equilibrium allows us to conduct meaningful comparative static analysis. The first exercise deals with the effect of a change in the number of students. Intuitively, an increase in the number of competitors (students) should decrease the probability of enrolling in any college leading students to decrease their investment in signaling. However, our results show that this effect is not symmetric across all students: in the face of a new competitor, low-skilled students' probability of enrolling in college decreases while high-skilled students' increases. Thus, the low-skilled students do reduce their investment in signaling but high-skilled ones actually increase it. We also analyze the effect of a change in the number of school seats and a change in college quality. The implications are similar. In particular, we show that an increase in the number of school seats or college quality leads low-skilled students to increase their investment in signaling and high-skilled students to decrease it.

The previous results are useful to explain an interesting empirical observation in real-world college admissions.[8] In the USA a decline in the mean SAT score has been observed as the participation rate increases (Wallace, Sep. 27, 2011). According to the College Board,[9] the mean SAT score has declined because more students of heterogeneous academic backgrounds are represented in the pool of test-takers, i.e. an increase in the number of applicants systematically decreases the proportion of good test-takers in the population. Our results suggest an alternative explanation based on the underlying signaling game of the problem.

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[8] California Postsecondary Education Commission (CPEC), "SAT Scores and Participation Rate" at http://www.cpec.ca.gov/StudentData/50StateSATScores.asp.

[9] "43% of 2011 College-Bound Senior Met SAT College and Career Readiness Benchmark" at http://press.collegeboard.org/releases/2011/43-percent-2011-college-bound-seniors-met-sat-college-and-career-readiness-benchmark

According to our model, an increase in the number of competitors not only leads low-skilled students to decrease their investment in signaling, but also increases the proportion of students who optimally decide to do so. Then, an increase in the number of competitors will eventually lead students to reduce the average investment in signaling with no change in the underlying distribution of student skills due to the increased competition and the presence of private information about student abilities.

Finally, we analyze the gains of the CSM, which are defined in a natural way as the difference in equilibrium payoffs between this separating signaling equilibrium of the CSM and a pooling equilibrium of the problem with no signaling. We show that students' gains are strictly increasing with respect to their skills. However, this property of students' gains does not guarantee that potential losses can be avoided. In addition, it is possible to show that, under certain distributions of skills, all students can have negative gains. Colleges' gains depend on the expected values of the order statistics of the distribution of student skills. Thus, the analysis of these gains is a difficult issue, since for most distributions there are no closed-form solutions for moments of order statistics. We analyze the particular case of exponentially distributed skills that allow for calculating a closed-form solution for colleges' gains. Even if the exponential model is a very peculiar case, it has interesting implications. First, colleges' gains are monotonically increasing in college quality, i.e. the best colleges have the greatest gains. Second, colleges' gains are monotonically increasing in the number of students, i.e. all colleges benefit from an increased demand for school seats. Finally, we show that a sufficiently large demand for school seats earns all colleges positive gains.

The rest of the paper is organized as follows. In Section 2, we describe the basic model and provide definitions. In Section 3, we analyze the benchmark college admissions problem with no signaling. In Section 4, we introduce the CSM and its equilibrium characterization. In Section 5, we conduct our comparative statics analysis. In Section 6, we analyze the gains of the CSM. In Section 7, we offer some concluding remarks. All proofs are presented in the Appendix.

2 The Model

There are $M \geq 1$ colleges and $N > M$ students. Let $C = \{c_1, c_2, ..., c_M\}$ denote the set of colleges with typical agent $c \in C$ and let $S = \{s_1, s_2, ..., s_N\}$ denote the set of students with typical agent $s \in S$. Each college $c \in C$ is characterized by an observable parameter $v_c > 0$, which is interpreted as its quality. With some abuse of notation, we use v_j to denote the quality of the college c_j. In order to simplify, we usually say "the student i" instead of "the student s_i" and "the college j" instead of "the college c_j". We assume, without loss of generality (w.l.g), that colleges' quality satisfy the following condition, $v_1 \geq v_2 \geq ... \geq v_M$.

Each student $s \in S$ is characterized by a parameter $\alpha_s \in [0, w]$ for any $w > 0$ that is interpreted as his skills or academic abilities. We say that a student s is more skilled than a student s' whenever $\alpha_s > \alpha_{s'}$. Students' skills are private information, this implies that only the student $s \in S$ knows the realization of his own parameter α_s. We assume that student skills are independently and identically distributed on some interval $[0, w]$ according to a strictly increasing and continuously differentiable cumulative distribution function F such that $F(0) = 0$ and $F(w) = 1$.[10] The distribution F has a continuous density $f = F'$ that satisfies $f(\alpha) > 0$ for all $\alpha \in (0, w)$. All elements of the model are common knowledge, i.e. the distribution of skills F; the number of students and colleges; and colleges' quality.

2.1 The Matching Problem

For simplicity, we focus on the simplest one-to-one matching problem,[11] i.e. each college has only one available school seat. In this setting, an assignment is a matching between colleges and students which is a mapping that specifies a partner for each agent, allowing for the possibility that some agents remain unmatched. Formally,

Definition 1 *A matching μ is a mapping from the set $S \cup C$ onto itself such that:*

[10] All results hold when students' parameters are independently and identically distributed on the interval $[0, \infty)$ according to a strictly increasing and continuously differentiable cumulative distribution function F such that $F(0) = 0$ and $\lim F(w) = 1$ as $w \rightarrow \infty$.

[11] The model can be easily extended to many-to-one matching problems under the assumption that colleges have responsive preferences (Roth and Sotomayor, 1990).

1. *If* $\mu(s) \neq s$ *then* $\mu(s) \in C$*;*

2. *If* $\mu(c) \neq c$ *then* $\mu(c) \in S$*; and*

3. $\mu(s) = c$ *if and only if* $s = \mu(c)$*.*

According to this definition, a student (college) with no partner is matched with himself (itself). In order to simplify, each student (college) gets a utility equal to the quality (skills) of the partner. Let $U_s(\mu)$ and $U_c(\mu)$ be the utilities of the student s and the college c, respectively, under the matching μ. Then each student $s \in S$ has a payoff,

$$U_s(\mu) = \begin{cases} v_c \text{ if } \mu(s) = c \\ 0 \;\; \text{Otherwise} \end{cases} \tag{1}$$

Each college $c \in C$ has a payoff,

$$U_c(\mu) = \begin{cases} \alpha_s \text{ if } \mu(c) = s \\ 0 \;\; \text{Otherwise} \end{cases} \tag{2}$$

We normalize the utility of remaining unmatched to zero for both colleges and students. In the two-sided matching literature, a college admissions problem is described by a three-tuple $(S, C, \succ)$, where S is a set of students, C is a set of colleges and $\succ = (\succ_{s_1}, ..., \succ_{s_N}; \succ_{c_1}, ..., \succ_{c_M})$ denotes a profile of ordinal preferences. In this setting, each agent $a \in S \cup C$ has a complete, strict and transitive preference relation $\succ_a$ over the set of agents on the opposite side of the market and the prospect of remaining unmatched.

It is easy to see that each student $s \in S$ has a preference relation $\succ_s$ over the set of colleges and the prospect of remaining unmatched $C \cup \{s\}$, such that: a) $c \succ_s s$ if and only if $v_c > 0$ and b) for all $c, c' \in C$, it is satisfied that $c \succ_s c'$ if and only if $v_c > v_{c'}$. Since college quality are observable, all students have identical ordinal preferences. In a similar way, each college $c \in C$ has a preference relation $\succ_c$ on the set of students and the prospect of having its position unfilled $S \cup \{c\}$, such that: a) $s \succ_c c$ if and only if $\alpha_s > 0$ and b) for all $s, s' \in S$, it is satisfied that $s \succ_c s'$ if and only if $\alpha_s > \alpha_{s'}$. Let $\succeq_a$ denote the weak preference relation associated with $\succ_a$ for each agent $a \in S \cup C$. Thus, for any $c, c' \in C$, $c \succeq_s c'$ implies either

$c \succ_s c'$ or $v_c = v_{c'}$. In a similar way, for any $s, s' \in S$, $s \succeq_c s'$ implies either $s \succ_c s'$ or $\alpha_c = \alpha_{c'}$.

A matching μ is **individually rational** whenever $\mu(a) \succeq_a a$ for all $a \in S \cup C$. A student-college pair (s, c) such that $\mu(s) \neq c$ **blocks** the matching μ if, $s \succ_c \mu(c)$ and $c \succ_s \mu(s)$. A matching μ is **stable** if it is individually rational and not blocked by any student-college pair. Let $\mathcal{E}(S, C, \succ)$ denote the set of **stable matchings** of the college admission problem $(S, C, \succ)$.

3 The Benchmark Problem: College Admissions with no Signaling

We analyze a benchmark problem of college admission with no signaling and incomplete information about student skills. In this setting, all students are ex-ante identical, since colleges only know the prior distribution of student skills but not their realizations. So the expected value of student abilities $E[\alpha]$ is the best estimation of student skills.

We consider that colleges and students are matched according to the following simple decentralized matching mechanism in two stages.

1. **Offers:** Each college $c \in C$ sends one message $m(c) \in S \cup \{c\}$. If $m(c) = s$, then the college c is making an offer to the student s. If $m(c) = c$, the college c is making no offer. Let $O(s) = \{c \in C : m(c) = s\} \cup \{s\}$ be the set of offers to the student s (note that a student always receives an offer from himself) ;

2. **Hiring:** Each student $s \in S$ chooses one of his available offers in $O(s)$.

Colleges and students play the game induced by this simple mechanism. In complete information environments with strict preferences, Alcalde and Romero-Medina (2000) show that this mechanism implements in **Subgame Perfect Equilibrium (SPE)** the set of stable matchings of college admissions problems. Thus this class of decentralized matching mechanisms exhausts the possibility of matching colleges and students in a stable way.[12] Further, under

[12]These results hold in problems where colleges have quotas of students providing colleges' preferences are responsive, see Roth and Sotomayor (1991).

certain conditions on agents' preferences,[13] this mechanism also maximizes the number of potential matches.

In the presence of incomplete information, these results do not hold any more. The mechanism may have many equilibria depending on the available information about student abilities and the degree of coordination among colleges. In this section, we focus on two "natural" equilibria of the problem, with and without coordination among colleges, whose characterization allows for analyzing the effects of the presence of incomplete information and the problem of coordination in decentralized college admissions. The explicit characterization of agents' payoffs is based on the number of students and colleges, the college quality, and the student skills and allows for identifying the effect of a change in one of these parameters on the equilibrium payoffs.

Before analyzing these equilibria, we argue that it is easy to characterize the equilibrium students' behavior. Since college quality is observable, in any possible equilibrium students must choose the best offer among the available ones. It is clear that under any alternative choice rule, students cannot get a better assignment. Then, a rule according to which students choose the best offer among the available ones is a dominant strategy. We assume that colleges anticipate optimal students' behavior and decide on their offers. For simplicity, we label each student with a number from 1 to N, these labels are observable for all agents and do not provide information about student skills.

We analyze an equilibrium situation in which colleges coordinate their actions based on student labels. Consider a profile of strategies where all students follow their dominant strategy while each college c_j sends one message to student j. Let μ be the outcome matching of this strategy profile. It is easy to verify that this assignment satisfies $\mu\left(c_j\right) = j$ for $j = 1, \ldots, M$ while the rest of students remain unmatched, i.e. $\mu(j) = j$ for $j = M + 1, \ldots, N$. Under this assignment, each college gets a payoff equal to $E[\alpha]$ while students get a payoff equal to v_j for $j = 1, \ldots, M$ and zero otherwise. It can be shown that this profile of strategies is a SPE

[13] When colleges have responsive preferences respect to individual preferences, the set of agent unmatched and unfilled positions are the same at any stable matching (Roth and Sotomayor, 1990). This result implies that this simple matching mechanism not only exhausts the possibility of matching agents in a stable way, but also maximizes the number of potential matches when colleges have responsive preferences.

of this college admissions game. First, note that no student has a profitable deviation, since students are following their dominant strategy. Second, a college c_k can deviate by sending a message to any alternative student $j \neq k$. In this case, the college c_k either will get matched with another student $j = k+1, ..., N$ or will be rejected by another student $j = 1, ..., k-1$. It is clear that this deviation cannot be profitable, since all students are ex-ante identical. Further, note that, under this equilibrium, the number of potential matches is maximized. This equilibrium is well defined for any permutation of the set of students and the equilibria stemming from the permutations are payoff equivalent for colleges.

Now we consider an equilibrium of this game with no coordination among colleges. We want to show that the profile of strategies where students choose the best offer among the available ones and colleges make one offer to each student with equal probability is a SPE of this college admissions game.

We consider a college admissions problem with $M \geq 1$ colleges and $N > M$ students. As before, we label each student with one number from 1 to N with no information about student skills but the prior distribution. Assume that each college sends one offer to each student with equal probability (i.e. $\frac{1}{N}$), we want to show that no college has a profitable deviation from this strategy. Consider that any college c_j with observable quality v_j is planning to deviate from this strategy. Note that there are $j-1$ colleges with higher quality and $M-j$ colleges with lower quality than college c_j. Since college quality is observable, an offer from college c_j always beats any other offer from the $M-j$ lower-quality colleges. Then, one offer from college c_j will be accepted by student i whenever every of the $j-1$ higher-quality colleges make one offer to any of the other $N-1$ students.

Thus, the total number of combinations of offers from $M-j$ colleges to N students is N^{M-j} while the total number of combinations of offers from $j-1$ colleges to $N-1$ students is $(N-1)^{j-1}$. Since colleges do not coordinate, one offer from college c_j will be accepted by student i with probability:

$$\frac{(N-1)^{j-1}\, N^{M-j}}{N^{M-1}} = \left(\frac{N-1}{N}\right)^{j-1} \quad \text{for } j = 1, ..., M. \tag{3}$$

Then, by making an offer to student i with probability $\frac{1}{N}$, college c_j will get an expected payoff of $\left(\frac{N-1}{N}\right)^{j-1} E[\alpha]$. Note that this payoffs only depends on the expected value of students skills, since all students are ex-ante identical.

Now consider that college c_j is planning to deviate from this strategy by making one offer to each student i with probability $p_i \neq \frac{1}{N}$. It is easy to see that such deviation cannot be profitable, since $\sum_{i=1}^{N} p_i \left(\frac{N-1}{N}\right)^{j-1} E[\alpha] = \left(\frac{N-1}{N}\right)^{j-1} E[\alpha]$ for any $p_i \neq \frac{1}{N}$ such that $\sum_{i=1}^{N} p_i = 1$. Then, the profile of strategies where colleges send one offer to each student with equal probability is a symmetric SPE of this game. Note that in this equilibrium colleges' payoffs EQ_{c_j} depend on the number of students, the expected value of student skills and the rank of colleges

$$EQ_{c_j} = \left(\frac{N-1}{N}\right)^{j-1} E[\alpha] \text{ for } j = 1, ..., M. \tag{4}$$

Next, we deduce students' payoffs in this symmetric equilibrium. In this case, we have to find the probability that each student $i = 1, ..., N$ enrolls in college c_j for $j = 1, ..., M$. First, we know that student i will reject any available offer but the best one. This implies that student i enrolls in college c_1 with probability $\frac{1}{N}$. It is easy to show that, in general, student i enrolls in college c_j with probability, $\frac{1}{N}\left(\frac{N-1}{N}\right)^{j-1}$. Then, the expected payoff of each student $i = 1, ..., N$ is given by

$$EU(N, M) = \frac{1}{N} \sum_{k=1}^{M} v_k \left(\frac{N-1}{N}\right)^{k-1} \tag{5}$$

Since students enroll in each college with positive probability, the students' payoffs are strictly positive for any $M \geq 1$ and $N > M$ and satisfy $v_1 > EU(N, M) > 0$. In addition, students may remain unmatched with a probability equal to $1 - \frac{1}{N} \sum_{k=1}^{M} \left(\frac{N-1}{N}\right)^{k-1} = \left(\frac{N-1}{N}\right)^{M}$, which is strictly positive, increasing in the number of students, and decreasing in the number of school seats available.

This simple model is useful to analyze the main consequences of the absence of coordination in college admissions with incomplete information. First, note that, for any number of students N and school seats M, all agents but the highest-quality college remain unmatched with positive probability. College c_1 fills its vacancy with probability one and gets an expected payoff equal to $E[\alpha]$ which is the best prediction of student skills without additional information. Second, the equilibrium assignment may be inefficient, since colleges only know the expected value of student skills. Further, the probability of enrolling a students is decreasing in the college rank, since the probability $\left(\frac{N-1}{N}\right)^{j-1}$ is strictly decreasing in j. Therefore, the absence of coordination mainly hurts low-quality colleges.

4 The Costly Signaling Mechanism

In this section, we analyze a decentralized matching mechanism called **Costly Signaling Mechanism** (CSM). Under this mechanism, each student $s \in S$ chooses a costly observable score $P_s \geq 0$ to signal his skills. Hence, a student $s \in S$ with type α who chooses a score P_s has to pay the cost

$$C(\alpha, P_s) = \frac{c(P_s)}{\phi(\alpha)} \tag{6}$$

We assume that the function $c(\cdot)$ is strictly increasing, continuously differentiable, and convex such that $c(0) = 0$. We also consider that the function $\phi(\cdot)$ is strictly increasing, continuous differentiable, and bounded in the interval $[0, w]$ such that $\phi(0) > 0$.

The profile of student scores $(P_s)_{s\in S}$ is observable for all agents. Under the CSM, colleges and students are match according to the following decentralized matching procedure in two stages:

1. **Signaling Stage:** Each student $s \in S$ with parameter α chooses a score $P_s \geq 0$ at the cost $C(\alpha, P_s)$.

2. **Matching Stage:** After observing the profile of scores $(P_s)_{s\in S}$, students and colleges match according to the following decentralized matching process:

   (a) **Offers:** Each college $c \in C$ sends one message $m(c) \in S \cup \{c\}$. If $m(c) = s$, college c is making an offer to student s. If $m(c) = c$, college c is making no offer. Let $O(s) = \{c \in C : m(c) = s\} \cup \{s\}$ be the set of offers of student s (a student always receives an offer from himself) ;

(b) **Hiring:** Each student $s \in S$ chooses one of his available offers $O(s)$.

We want to characterize a symmetric and strictly separating equilibrium where all students use the same signaling strategy. Obviously, the model can admit many other symmetric equilibria. For instance, pooling equilibria where no student invests in signaling (in this situation any of the symmetric equilibria analyzed in the previous section can be sustained) or semi-separating equilibria.

To sustain the strictly separating equilibrium, we consider beliefs according to which students with higher scores are associated with higher academic skills. Formally, we describe these beliefs by a continuous distribution of student abilities given the score $P > 0$, i.e. a continuous cumulative distribution $G(\alpha \mid P)$. We assume that these beliefs have a continuous density $g(\alpha \mid P)$ and satisfy $G(\alpha \mid P') < G(\alpha \mid P)$ for all $\alpha \in (0, w)$ whenever $P' > P$. Note that these conditions imply that $E[\alpha \mid P'] > E[\alpha \mid P]$ for all $\alpha \in (0, w)$ whenever $P' > P$ where $\int \alpha g(\alpha \mid P)\, d\alpha = E[\alpha \mid P]$. We consider that after observing the profile of student scores $(P_s)_{s \in S}$, colleges "update" their ordinal preferences in the following simple way. Each college $c \in C$ forms an auxiliary preference relation $\succ_c^*$ over the set of students and the prospect of remaining unmatched, $S \cup \{c\}$ such that: a) $s \succ_c^* c$ if and only if $P_s > 0$ and b) for any $s, s' \in S$, $s \succ_c^* s'$ if and only if $P_s > P_{s'}$. Note that the profile of interim preferences $\succ_C^* = \left(\succ_c^*\right)_{c \in C}$ is consistent with beliefs $G(\alpha \mid P)$. Thus, students with higher scores are associated with higher expected skills. Under these conditions, we can establish the following result.

Proposition 1 *Consider a set of beliefs $G(\alpha \mid P)$ such that $G(\alpha \mid P') < G(\alpha \mid P)$ for all $\alpha \in (0, w)$ whenever $P' > P$ and assume that these beliefs have a continuous density $g(\alpha \mid P) > 0$. Then, for any profile of student scores $(P_s)_{s \in S}$ such that $P_s \neq P_{s'}$ for all $s, s' \in S$ and $s \neq s'$, there is a unique SPE outcome in the second stage of the CSM. This equilibrium outcome is the unique stable matching of a college admissions problem, $\left(S, C, \left(\succ_S, \succ_C^*\right)\right)$. Further, this unique equilibrium assignment is assortative.*

According to the previous proposition, only stable matches between students and colleges are reasonable outcomes of the CSM (Alcalde and Romero-Medina, 1998). Further, under the interim college preferences $\succ^*_C = \left(\succ^*_c\right)_{c \in C}$ the outcome is assortative, i.e. the highest-scored student is matched with the best college; the second highest-scored student is matched with the second best college, and so on.

In the following section, we analyze the signaling equilibrium of the first stage of the CSM. We focus on a symmetric pure strategy equilibrium where all students play according to the same signaling strategy. We analyze a settings with $M \geq 1$ school seats and $N > M$ students. However, the model can be easily extended to analyze any problem with the same number of students and school seats.

4.1 The Signaling Equilibrium

We analyze the signaling equilibrium of the first stage of the CSM. To characterize this equilibrium, we take the outcome of the matching stage of the mechanism as given. To illustrate the problem, we focus on the simplest case with only one college with quality $v_1 > 0$ and $N > 1$ students.

In this setting, we analyze a separating symmetric Nash equilibrium of the signaling game played by students. This equilibrium is characterized by a continuously differentiable and strictly increasing signaling strategy with respect to student abilities. We focus on student 1's problem, who chooses a score P_1 to signal his skills, while the rest of the students play according to the signaling strategy $\rho : [0, w] \to \Re_+$ which is assumed to be strictly increasing and continuously differentiable in α such that $\rho(0) = 0$.

Since the outcome matching of the CSM is assortative, student 1 with parameter α gets the payoff $v_1 - C(\alpha, P_1)$ whenever $P_1 > \rho(\alpha_i)$ for all $i \neq 1$. This occurs with probability $\Pr[P_1 > \rho(\alpha_2), ..., P_1 > \rho(\alpha_N)] = F\left(\rho^{-1}(P_1)\right)^{N-1}$ given that student skills are identically and independently distributed according to F. Otherwise, student 1 gets the payoff $-C(\alpha, P_1)$ with probability $1 - F\left(\rho^{-1}(P_1)\right)^{N-1}$. Hence, the expected payoff of student 1 with parameter

α, when the rest of the students play according to the signaling function $\rho(\cdot)$, is:

$$\pi(\alpha, P_1) = v_1 F\left(\rho^{-1}(P_1)\right)^{N-1} - C(\alpha, P_1) \tag{7}$$

Student 1 takes as given the signaling strategy of the rest of the students and chooses a score P_1 to maximize his expected payoff $\pi(\alpha, P_1)$. The first order condition (FOC) with respect to P_1 leads to the following condition,

$$v_1(N-1)F\left(\rho^{-1}(P_1)\right)^{N-2} f\left(\rho^{-1}(P_1)\right)\frac{1}{\rho'\left(\rho^{-1}(P_1)\right)} - \frac{c'(P_1)}{\phi(\alpha)} = 0 \tag{8}$$

By reordering the previous expression, we obtain the following differential equation,

$$v_1(N-1)\phi(\alpha)F\left(\rho^{-1}(P_1)\right)^{N-2} f\left(\rho^{-1}(P_1)\right) = c'(P_1)\rho'\left(\rho^{-1}(P_1)\right) \tag{9}$$

In a symmetric equilibrium $P_1 = \rho(\alpha)$, then the previous differential equation becomes:

$$v_1(N-1)\phi(\alpha)F(\alpha)^{N-2} f(\alpha) = c'(\rho(\alpha))\rho'(\alpha) \tag{10}$$

By solving this differential equation with the initial condition $\rho(0) = 0$, we find that the equilibrium signaling strategy satisfies

$$\rho(\alpha) = c^{-1}\left(v_1(N-1)\int_0^\alpha \phi(x)F(x)^{N-2} f(x)\,dx\right) \tag{11}$$

The equilibrium signaling strategy $\rho(\cdot)$ is strictly increasing and continuously differentiable in α. Note that $\rho(\cdot)$ only satisfies the FOC of student 1's maximization problem, which is necessary but not sufficient to characterize the signaling equilibrium. Hence, we have to prove that the signaling strategy $\rho(\cdot)$ is in fact a symmetric equilibrium of this game. The equilibrium payoff of any student with parameter α is given by,

$$\pi(\alpha, \rho(\alpha)) = v_1 F(\alpha)^{N-1} - \frac{c(\rho(\alpha))}{\phi(\alpha)} \tag{12}$$

It is not difficult to show that this payoff function satisfies $\frac{d}{d\alpha}\left(\pi\left(\alpha,\rho\left(\alpha\right)\right)\right) > 0$ and $\pi\left(0,\rho\left(0\right)\right) = 0$. Hence, we show that any alternative score $P' \neq \rho\left(\alpha\right)$ cannot be a profitable deviation for any student with parameter α. Consider that a student with parameter α is planning to choose another score $0 < P' < \rho\left(\alpha\right)$ while the rest of the students are playing according to the signaling strategy $\rho\left(\alpha\right)$. Since the signaling strategy is strictly increasing in α and satisfies $\rho\left(0\right) = 0$, there exists a unique $0 < \alpha' < \alpha$ such that $\rho\left(\alpha'\right) = P'$. This implies that a student who chooses an alternative strategy $P' = \rho\left(\alpha'\right)$ will get a payoff of $\pi\left(\alpha, P'\right) = \pi\left(\alpha,\rho\left(\alpha'\right)\right)$. Hence, a student with parameter α loses the extra payoff $\pi\left(\alpha,\rho\left(\alpha\right)\right) - \pi\left(\alpha,\rho\left(\alpha'\right)\right)$ by deviating to $\rho\left(\alpha'\right) = P'$. Then

$$\pi\left(\alpha,\rho\left(\alpha\right)\right) - \pi\left(\alpha,\rho\left(\alpha'\right)\right) = v_1\left(F\left(\alpha\right)^{N-1} - F\left(\alpha'\right)^{N-1}\right) - \frac{c\left(\rho\left(\alpha\right)\right) - c\left(\rho\left(\alpha'\right)\right)}{\phi\left(\alpha\right)} \tag{13}$$

The extra payoffs $\pi\left(\alpha,\rho\left(\alpha\right)\right) - \pi\left(\alpha,\rho\left(\alpha'\right)\right)$ can be reduced to

$$v_1\left(F\left(\alpha\right)^{N-1} - F\left(\alpha'\right)^{N-1}\right) - \frac{1}{\phi\left(\alpha\right)}v_1\left(N-1\right)\int_{\alpha'}^{\alpha}\phi\left(x\right)F\left(x\right)^{N-2}f\left(x\right)dx \tag{14}$$

Since the function $\phi\left(x\right)$ is positive, strictly increasing in x, and bounded in $[0,w]$, it is clear that the following inequality holds

$$\frac{1}{\phi\left(\alpha\right)}v_1\left(N-1\right)\int_{\alpha'}^{\alpha}\phi\left(x\right)F\left(x\right)^{N-2}f\left(x\right)dx < v_1\int_{\alpha'}^{\alpha}\left(N-1\right)F\left(x\right)^{N-2}f\left(x\right)dx \tag{15}$$

In addition, note that, by definition, $\int_{\alpha'}^{\alpha}\left(N-1\right)F\left(x\right)^{N-2}f\left(x\right)dx = F\left(\alpha\right)^{N-1} - F\left(\alpha'\right)^{N-1}$. This last condition implies that $\pi\left(\alpha,\rho\left(\alpha\right)\right) - \pi\left(\alpha,\rho\left(\alpha'\right)\right) > 0$ for all $\alpha' < \alpha$ which proves that $P' = \rho\left(\alpha'\right)$ is not a profitable deviation. Following a similar argument, it is possible to show that any alternative score $P'' > \rho\left(\alpha\right)$ cannot be a profitable deviation either. Then the signaling strategy $\rho\left(\alpha\right)$ is a symmetric equilibrium of the signaling game played by students. In the following section, we show that all of these results hold in the general case with $M \geq 1$ colleges and $N > M$ students. All proofs and calculations can be found in the Appendix.

4.1.1 The General Case: $N > M \geq 1$

Now consider a general case with N students and M colleges such that $N > M \geq 1$. Assume w.l.g. that all colleges have different quality and satisfy $v_1 > v_2 > ... > v_M > 0$. As before, we analyze student 1's maximization problem with parameter $\alpha \in (0, w)$ while all the other students play according to some signaling strategy $\rho_M : [0, \infty) \to \Re_+$. As before, we assume that the signaling strategy $\rho_M(\cdot)$ is strictly increasing and continuously differentiable in α such that $\rho_M(0) = 0$.

Student 1 chooses a score $P_1 \geq 0$ to signal his abilities. We say that student 1 has a "success" whenever $P_1 > \rho_M(\alpha_i)$ for some other student $i \neq 1$ and a "failure" whenever $P_1 < \rho_M(\alpha_i)$ for some other student $i \neq 1$. The probability of having one "success" is $F\left(\rho_M^{-1}(P_1)\right)$ whereas the probability of having one "failure" is $1 - F\left(\rho_M^{-1}(P_1)\right)$. Note that these probabilities are independent, since students' parameters are independently distributed.

For any given number of students $N > M$, student 1 with score P_1 enrolls in college c_j with quality v_j, whenever he has $N - j$ "successes" and $j - 1$ "failures". Note that this situation happens $\binom{N-1}{j-1}$ different times, then the probability of enrolling in college c_j is

$$\binom{N-1}{j-1} F\left(\rho_M^{-1}(P_1)\right)^{N-j} \left[1 - F\left(\rho_M^{-1}(P_1)\right)\right]^{j-1} \tag{16}$$

The previous argument implies that the probability of enrolling in college $c_j \in C$ follows a binomial distribution. Hence, the expected payoff of student 1, $\pi(\alpha, P_1)$, satisfies

$$\pi(\alpha, P_1) = \sum_{k=1}^{M} v_k \binom{N-1}{k-1} F\left(\rho_M^{-1}(P_1)\right)^{N-k} \left[1 - F\left(\rho_M^{-1}(P_1)\right)\right]^{k-1} - C(\alpha, P_1) \tag{17}$$

Student 1 takes as given the signaling strategy of the rest of the students and chooses a score P_1 to maximize his expected payoff $\pi(\alpha, P_1)$. In Appendix A, we solve student 1's maximization problem in a symmetric equilibrium where all students play according to the same signaling strategy $\rho_M(\alpha)$. We show that the signaling function that satisfies the FOC of student 1's maximization problem characterizes this symmetric separating equilibrium. Thus, we establish the following result.

Proposition 2 *The signaling strategy,*

$$\rho_M(\alpha) = c^{-1}\left(\sum_{k=1}^{M-1}(v_k - v_{k+1})\int_0^{\alpha}\phi(x)\, f_{(k,N-1)}(x)\,dx + v_M\int_0^{\alpha}\phi(x)\, f_{(M,N-1)}(x)\,dx\right)$$

is a symmetric equilibrium of college admissions problems with $M \geq 1$ *colleges and* $N > M$ *students.*

Proof. See Appendix A. ■

Given the equilibrium signaling strategy $\rho_M(\cdot)$, a student with parameter α will get expected payoff

$$\pi(\alpha, \rho_M(\alpha)) = \sum_{k=1}^{M} v_k \binom{N-1}{k-1} F(\alpha)^{N-k}\left[1 - F(\alpha)\right]^{k-1} - \frac{c(\rho_M(\alpha))}{\phi(\alpha)} \tag{18}$$

Note that to characterize the signaling equilibrium, we assume some desirable properties of the equilibrium signaling strategy. We should show that these properties are satisfied in equilibrium. A simple observation is enough to show that the equilibrium signaling strategy and agents' payoffs are continuously differentiable functions in α. In the following proposition we establish some interesting properties of the signaling strategy and equilibrium payoffs.

Proposition 3 *The equilibrium signaling strategy* $\rho_M(\alpha)$ *and student's payoff* $\pi(\alpha, \rho_M(\alpha))$ *satisfy the following properties:*

1. $\rho_M(\alpha)$ *is strictly increasing in* α *and bounded from above.*

2. $\pi(\alpha, \rho_M(\alpha))$ *is strictly increasing in* α.

Proof. See Appendix A. ■

Since the equilibrium signaling strategy is strictly increasing and the probability of having two students with the same skills is zero, no pair of students will choose the same score. Then, the equilibrium outcome of the CSM will be assortative with respect to the true student skills.

Hence, the highest-skilled student will be matched with the best college, the second highest-skilled student will be matched with the second best college, and so on. Further, students with higher abilities will get greater payoffs. This result comes from the single crossing property of the signaling cost function, since higher-skilled students have lower marginal signaling cost.

On the other hand, the assortative structure of the equilibrium assignment of the CSM implies that colleges' payoffs depend on the ranking of the enrollees and the prior distribution of student skills. Let μ^* be the unique equilibrium outcome of the **CSM**, then, in equilibrium, colleges get expected payoffs

$$EQ^*_{c_j} = E\left[\alpha \mid P_{\mu^*(c)}\right] = E\left[\alpha_{(j,N)}\right] = \int_0^w \alpha f_{(j,N)}(\alpha)\, d\alpha \text{ for } j = 1, ..., M. \tag{19}$$

Where $\alpha_{(j,N)}$ is the j-th order statistic from a sample of size N such that $\alpha_{(1)} = \max_{1\leq i\leq N} \alpha_i$, $\alpha_{(2)}$ = second greatest α_i, and so on. It is well known that the order statistic $\alpha_{(j,N)}$ is distributed according to the probability density function

$$f_{(j,N)}(\alpha) = \frac{N!}{(j-1)!\,(N-j)!} f(\alpha) F(\alpha)^{N-j} [1 - F(\alpha)]^{j-1} \text{ for } j = 1, ..., M. \tag{20}$$

Under these conditions, it is not difficult to show that responding to students' signals is a best response for colleges. Responding to students' signals implies ranking students based on their signals. First, it can be shown that the best strategy for any college c_j is to respond to students' signals, providing that all the colleges with higher quality than j, $\left\{c_1, c_2, ..., c_{j-1}\right\}$, do. The argument is very simple, college c_j has to compare the expected skills of enrollees between the responding to students' signals admission rule and any alternative admission rule. Note that college c_j knows that all the students are willing to accept its offer but those already enrolled in colleges $\left\{c_1, c_2, ..., c_{j-1}\right\}$, since, by assumption, those colleges respond to signals and have greater quality. This implies that any potential enrollee of the college c_j has skills $\alpha \leq \alpha_{(j)}$. By responding to students' signals, college c_j will enroll the best student among the available ones. In contrast, under any other admission rule, it will enroll a lower-skilled student.

Now consider the case of the best college, c_1, that knows that its offer will be accepted by any student. Since by responding to students' signals college c_1 will enroll the best student possible, it is optimal for c_1 to respond to students' signals. Then, a simple induction argument shows that all colleges will respond to students' signals.

5 Comparative Statics

In the previous sections, we characterized a separating symmetric equilibrium of the signaling game induced by the CSM. This equilibrium is characterized by a signaling strategy that depends on several parameters, such as the prior distribution of skills, the number of students, the number of school seats and the quality of colleges. Our explicit characterization allows for conducting a series of interesting comparative statics exercises to analyze the impact of a change in the underlying parameters of the model on the equilibrium signaling strategy. In particular, we focus on three simple exercises:

1. A change in the number of students;
2. A change in the number of school seats; and
3. A change in the quality of colleges.

The analysis of these exercises can guide our understanding of real-world college admissions as a signaling process whose outcome depends on the interaction of strategic decision makers. Furthermore, our model provides a good approach to analyze the effect of a change in the underlying parameters of the model.

One of the most important real-world signals in college admissions is the SAT test in the USA. Most students take the SAT during the last year of high school and almost all colleges and universities use that test scores for admission decisions. Empirical studies have analyzed the importance of the SAT and provided empirical support for our model of decentralized college admissions with incomplete information and costly signaling as a good approach for studying real-world college admissions. First, the SAT is arguably a costly signal that depends on the amount of effort applied by students. Second, it is well known that there is a strong positive correlation between SAT scores and student skills. For instance, Frey and Detterman (2004) show that there is a high correlation between SAT scores and several measures of student abilities, such as the IQ. Finally, the matching between colleges and students tends to be assortative with respect to the true student skills, since the best colleges and universities tend to enroll students with higher SAT scores (Webster, 2001a, 2001b).

It is clear that an incorrect understanding of the underlying signaling game in college admissions may lead to incorrect policy recommendations. For instance, empirical evidence in the USA shows that there is a decline in the average SAT score as the participation rate increases. If we only considered the high correlation between SAT scores and measures of student skills, we could suggest that the decline in SAT scores comes from an increase in the proportion of low-skilled students among test-takers, which may imply a change in the current distribution of student skills. According to this argument, a policy recommendation could be increasing the expenditure and duration of SAT coaching and tutorials in high school in order to improve the abilities of test-takers. However, the previous argument and policy recommendation ignore the underlying signaling game in college admissions, since they do not consider that, facing of new competitors, students may strategically decrease or increase their investment in signaling with no actual change in the underlying distribution of academic skills.

5.1 A Change in the Number of Students

In this section, we analyze the effect of a change in the number of students over the equilibrium signaling strategy of the CSM. Intuitively, an increase in competition for school seats may decrease the probability of enrolling in college, which leads students to decrease their investment in signaling. This intuition may seem correct at first; however, it does not consider that the effect of an increased competition on the enrollment probabilities may be asymmetric across students. For instance, in the presence of new students competing for a seat in college the probability of facing more qualified competitors is high for low-skilled students and low for high-skilled students. This asymmetry in the expected competition may lead students to react differently depending on their skills, thus generating an observed asymmetric effect on the signaling strategy.

In order to simplify and make our arguments as intuitive as possible, we consider a very simple setting with only one college with several school seats and an increasing number of students.[14] In this setting the equilibrium signaling strategy satisfies the following specification:

$$\rho_M(\alpha, N) = c^{-1}\left(v_1 \int_0^{\alpha} \phi(x)\, f_{(M,N-1)}(x)\, dx \right) \tag{21}$$

The result of this exercise shows that the effect on investment in signaling depends on academic abilities. Formally,

Proposition 4 *For any college admissions problem with one college with $M \geq 1$ school seats and $N > M$ students there exists a well-defined threshold $\alpha_N(N)$ such that the following holds:*

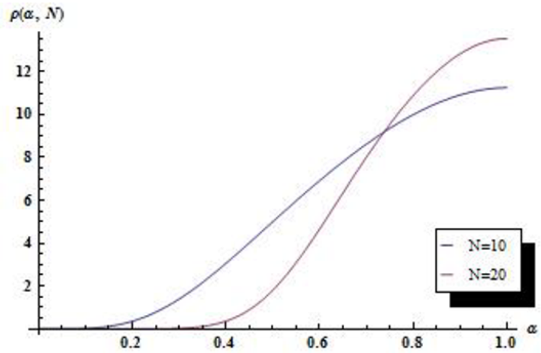
1. $\rho_M(\alpha, N+1) < \rho_M(\alpha, N)$ *for all* $\alpha < \alpha_N(N)$*;*
2. $\rho_M(\alpha, N+1) \geq \rho_M(\alpha, N)$ *for all* $\alpha \geq \alpha_N(N)$*; and*
3. $\alpha_N(N)$ *is strictly increasing in* N*.*

Proof. See Appendix B. ■

According to the previous result, we find that the low-qualified students decrease their investment in signaling while the high-skilled students increase it as the competition for school seats increases. A student with parameter α obtains a seat in college c_1 with probability

[14] In this particular case, we have a many-to-one matching problem where colleges have responsive preferences over the set of students. Under these conditions the proper concept of stability is group stability, (i.e. a matching is group stable if it is not blocked by any coalition of agents) instead of pairwise stability as in the one-to-one matching problem. However, it is easy to show that, under responsive preferences, a matching is group stable in a many-to-one matching problem if and only if it is pairwise stable in a properly related one-to-one matching problem (Roth and Sotomayor, 1991). Intuitively, a many-to-one matching problem with responsive preferences is equivalent to a one-to-one matching problem where each school seat is considered as a college with only one seat that has the same preferences over the set of students and each student uses an arbitrary and strict tie-breaking rule to order the school seats in the same college.

Figure 1: Effect of increasing the number of students

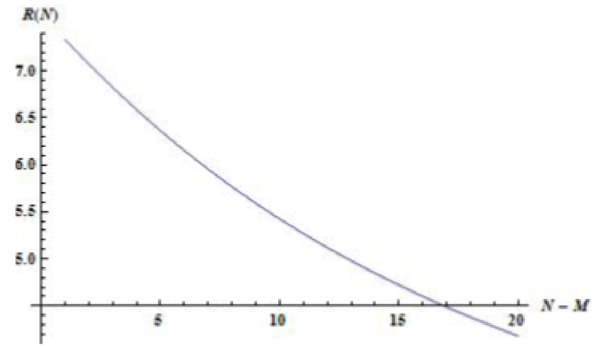


Source: Based on equilibrium signaling strategy simulations of the CSM for exponentially distributed student skills with comulative distribution $F(\alpha;\theta) = 1 - e^{-\frac{\alpha}{\theta}}$ and expected value $E[\alpha] = \theta$=1. For simplicity, we consider only one college with one school seat.

$\binom{N-1}{M-1} F(\alpha)^{N-M} [1 - F(\alpha)]^{M-1}$. In the presence of more competition for school seats, we can identify two opposite effects over the probability of enrollment. On the one hand, the probability of obtaining a school seat in college c_1 over $N - M$ students is reduced as N increases. As a consequence, students adjust their effort and reduce their investment in signaling. On the other hand, the number of draws $\binom{N-1}{M-1}$ where a student with parameter α must beats $N - M$ competitors increases as N increases. This implies that a student must increase his investment in signaling in order to beat those additional competitors and obtain a school seat in college c_1. Then, the previous result shows that the net effect between those two opposite effects depends on student skills. For bad students the net effect is negative while for good students it is positive.

A second interesting implication addresses the behavior of the average investment in signaling. Let $R(N) = \int \rho_M(\alpha, N) f(\alpha)\, d\alpha$ be the expected (average) investment in signaling. Our previous result shows that the effect of competition on $R(N)$ is ambiguous, since a fraction of students $\alpha_N(N)$ decreases their investment in signaling while the rest of students $w - \alpha_N(N)$ increases it in the presence of more students. However, the monotonicity of the threshold $\alpha_N(N)$ together with some properties of the equilibrium signaling strategy $\rho_M(\alpha, N)$ provide some insights about the behavior of $R(N)$ in the presence of a sufficiently

Figure 2: Average investment in signaling with respect to N



Source: Based on equilibrium signaling strategy simulations of the CSM for exponentially distributed student skills with comulative distribution $F(\alpha; \theta) = 1 - e^{-\frac{\alpha}{\theta}}$ and expected value $E[\alpha] = \theta$=1. For simplicity, we consider only one college with one school seat.

high competition for school seats.

First, the threshold $\alpha_N(N)$ is increasing in N, which implies that the mass of students $w - \alpha_N(N)$ who are willing to increase their investment in signaling in the presence of new competitors is decreasing and becomes a set of zero measure in the limit, as N tends to infinity. In addition, the monotonicity of $\alpha_N(N)$ also implies that a student who reduces his investment in signaling with N competitors, will never increase it with $N + 1$ competitors. Second, the equilibrium signaling strategy $\rho_M(\alpha, N)$ is increasing in α and bounded from above by an upper bound $\bar{\rho} = c^{-1}(v_1\phi(w))$ that is independent on N and M. This implies that students in the set $w - \alpha_N(N)$ cannot increase their investment in signaling above the upper bound $\bar{\rho}$. According to the previous argument, there must exist a sufficiently large demand for school seats such that the increase in the signal $\rho_M(\alpha, N)$ by $w - \alpha_N(N)$ students cannot compensate the decrease done by $\alpha_N(N)$ students leading to a fall of $R(N)$. Formally, there must exist a sufficiently large number of students $\hat{N}$ such that $R(N + 1) < R(N)$ for all $N \geq \hat{N}$.

The previous argument is consistent with an empirical observation in US college admissions. In the USA, the impact of increasing the number of test-takers on the mean SAT score has been extensively analyzed. According to data from the College Board for several years, a decline in the mean SAT score has been observed as the participation rate increased. The College Board explains this stylized fact in the following way:[15]

"*It is common for mean scores to decline slightly when the number of students taking an exam increases because more students of varied academic backgrounds are represented in the test-taking pool.*"

This interpretation implies that an increase in the number of applicants systematically decreases the proportion of good test-takers. By considering the positive and strong correlation between SAT scores and several measures of student abilities, this observation would also imply a change in the underlying distribution of students skills in the presence of more competition for school seats. However, this interpretation ignores the underlying signaling game in college admissions and the possibility that students strategically change their investment in signaling as a result of the presence of a greater number of competitors for the same number of school seats. Our model suggests that an increase in the number of applicants not only leads the low-skilled students to decrease their investment in signaling, as a response to the fall in the probability of being enrolled in college, but also decreases the proportion of good students who are willing to increase their investment in signaling in the face of more competition.

5.2 A Change in the Number of School Seats

The following exercise deals with the effect of a change in the number of school seats on the equilibrium signaling strategy. For simplicity, we consider a setting where the number of school seats can increase but remains lower than the number of students. Intuitively, an increase in the number of the available school seats should increase the probability of enrolling in college leading students to increase their investment in signaling.

Our model shows that this intuitive argument may not be correct, at least not for all students. The effect of a change in the number of school seats is not symmetric across students.

[15] College board (2011), "43% of 2011 College-Bound Senior Met SAT College and Career Readiness Benchmark" at http://press.collegeboard.org/releases/2011/43-percent-2011-college-bound-seniors-met-sat-college-and-career-readiness-benchmark

As in the previous section, we consider a very simple model with N students and one college that offers $M \geq 1$ school seats. In this setting, it is easy to show that the equilibrium signaling strategy of the problem satisfies the following expression

$$\rho_M(\alpha, M) = c^{-1}\left(v_1 \int_0^{\alpha} \phi(x) f_{(M,N-1)}(x)\, dx \right) \tag{22}$$

Then we establish the following result.

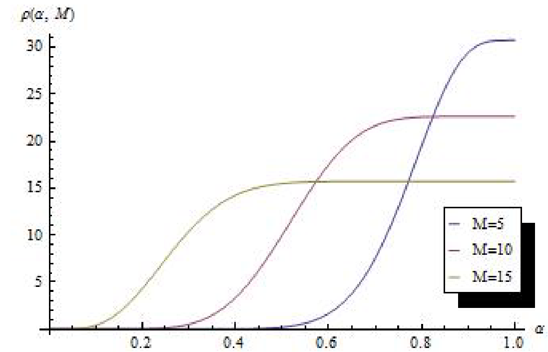
Proposition 5 *For any college admissions problem with one college with $M \geq 1$ school seats and $N > M$ students there exists a well-defined threshold $\alpha_M(M)$ such that the following holds:*

1. $\rho_M(\alpha, M+1) > \rho_M(\alpha, M)$ *for all* $\alpha < \alpha_M(M)$*;*
2. $\rho_M(\alpha, M+1) \leq \rho_M(\alpha, M)$ *for all* $\alpha \geq \alpha_M(M)$*; and*
3. $\alpha_M(M)$ *is strictly monotone decreasing in* M*.*

Proof. See Appendix B. ■

As in the previous result, a change in the number of school seats affects the probability of being accepted in college $\binom{N-1}{M-1} F(\alpha)^{N-M} [1 - F(\alpha)]^{M-1}$. An additional school seat has two effects on this probability. First, it increases the probability of having $N - M$ successes and decreases the probability of having $M - 1$ failures of being accepted in college, the net effect depends on student skills. We can say that a student with parameter α is good if $\frac{1-F(\alpha)}{F(\alpha)} < 1$, otherwise the students is bad. For good students the net effect of having an additional school seat is negative whereas for bad students it is positive. Second, an increase in the number of available seats in college affects the number of occurrences where a student is assigned a seat, i.e. $\binom{N-1}{M-1}$. In the presence of a new school seat, this number of occurrences increases if M is small relative to N and decreases otherwise. According to the previous argument, the net effect over the probability of enrolling in college depends on student skills and on the number of school seats available relative to the number of students. Our result shows that an increase in the number of school seats leads a low-skilled student to increase his investment

Figure 3: Effect of increasing the number of school seats M



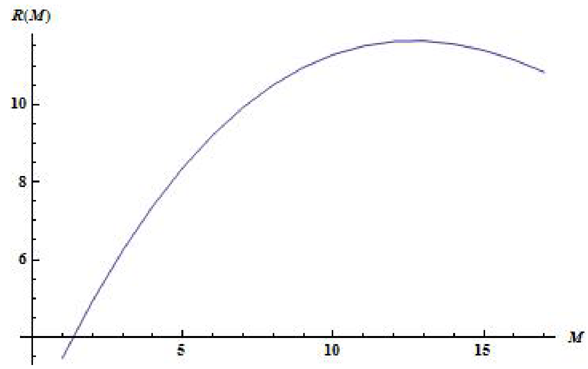
Source: Based on equilibrium signaling strategy simulations of the CSM for exponentially distributed student skills with comulative distribution $F(\alpha;\theta) = 1 - e^{-\frac{\alpha}{\theta}}$ and expected value $E[\alpha] = \theta$=1. For simplicity, we consider an example with N=20.

in signaling while a high-skilled student decreases it. Intuitively, an increase in the number of available seats in college could be equivalent to a decrease in competition for a fixed number of school seats.

The threshold $\alpha_M(M)$ is monotonically decreasing in M, i.e. $\alpha_M(M+1) < \alpha_M(M)$ for $M < N$. This result implies that the mass of students who are willing to increase their investment in signaling is reduced in the presence of more school seats whereas the proportion of students that decrease their investment in signaling becomes bigger. This property of the threshold $\alpha_M(M)$ allows for establishing some general conclusions about the shape of the average investment in signaling as a function of the number of school seats.

Since the effect of an additional school seat is not ambiguous for good and bad students and it is independent of the values of M and N, the result on the equilibrium signaling strategy depends mostly on the behavior of the binomial function $\binom{N-1}{M-1}$. As the number of school seats increases, the binomial function also increases until a maximum, after that an additional school seat decreases it. Intuitively, when M is small relative to N, an additional school seat is very valuable for students which leads even for some of the good students to increase their investment in signaling. In contrast, when the number of school seats is closer to the number of students, a newly available school seat is not as valuable as before and this leads even

Figure 4: Average investment in signaling with respect to M



Source: Based on equilibrium signaling strategy simulations of the CSM for exponentially distributed student skills with comulative distribution $F(\alpha;\theta) = 1 - e^{-\frac{\alpha}{\theta}}$ and expected value $E[\alpha] = \theta$=1. For simplicity, we consider an example with N=20.

for some of the bad students to decrease their investment in signaling. Thus, starting from a small number of seats in college, the average investment in signaling must increase until a big enough number of school seats is made available. Once this big enough number has been reached, any additional seat in college decreases the average investment in signaling.

5.3 A Change in the Quality of Colleges

In this section, we analyze the effect of a change in college quality on the equilibrium signaling strategy. We focus on a change in quality that preserves the ordinal preferences of students. For instance, if college c_k changes its quality from v_k to v'_k, it should be true that $v_{k-1} > v'_k > v_{k+1}$, whenever $v_{k-1} > v_k > v_{k+1}$. This assumption makes the equilibrium signaling strategies comparable before and after the change in college quality, since the equilibrium assignment of the CSM is the same in both situations.

Intuitively, when a college increases its quality the average quality of schools also increases, this increment in students' valuations makes it reasonable for students to increase their investment in signaling. However, as in the previous cases, this result depends on student abilities. Let $sgn(x)$ be a function such that $sgn(x) = 1$ if $x > 0$, $sgn(x) = -1$ if $x < 0$

and $sgn(x) = 0$ if $x = 0$. Then we establish the following result.

Proposition 6 *For any college admissions problem with $M \geq 2$ colleges and $N > M$ students there exists a well-defined threshold $\alpha_{v_k}(N,k)$ such that the following holds:*

1. $sgn\left(\rho_M\left(\alpha, v'_k\right) - \rho_M\left(\alpha, v_k\right)\right) = sgn\left(v'_k - v_k\right)$ *for all* $\alpha < \alpha_{v_k}(N,k)$ *and* $k = 1, ..., M$*;*

2. $sgn\left(\rho_M\left(\alpha, v'_k\right) - \rho_M\left(\alpha, v_k\right)\right) = -sgn\left(v'_k - v_k\right)$ *for all* $\alpha > \alpha_{v_k}(N,k)$ *and* $k = 1, ..., M$*; and*

3. $\alpha_{v_k}(N,k)$ *is monotone increasing* N *for all* $k = 2, ..., M$ *and monotone decreasing in* k.

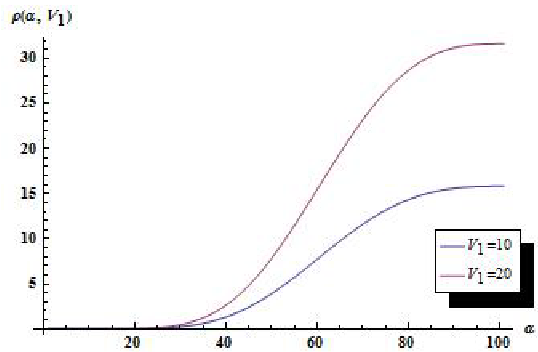
Proof. See Appendix B. ■

The previous result implies that the low-skilled and the high-skilled students change their investment in signaling in opposite direction in the presence of an increase of college quality. Intuitively, an increase in college quality for low-skilled students is more valuable than for high-skilled ones. We also show that the threshold $\alpha_{v_k}(N,k)$ is decreasing in k, i.e. $\alpha_{v_k}(N,k+1) < \alpha_{v_k}(N,k)$ for all $k = 1, ..., M-1$, which implies that students are more willing to increase their investment in signaling for a change in the quality of the high-quality colleges. Furthermore, in Appendix B, we show that $\alpha_{v_1}(N,1) = w$ for any N, which implies that only an increase in the quality of the college c_1 has no asymmetric effects across students as all students are willing to increase their investment in signaling.

6 Gains of the CSM

In the previous sections, we characterized two kind of equilibria of the CSM. A pooling equilibrium with no signaling and a strictly separating equilibrium with costly signaling. Both equilibria have several properties about equilibrium matches and agents' payoffs that allow for understanding the role of incomplete information and costly signaling in college admissions. In this section, we present a very simple comparison between both equilibria in order to verify whether the inclusion of costly signaling entails welfare gains or losses.

Figure 5: A change in the quality of the best college



Source: Based on equilibrium signaling strategy simulations of the CSM for exponentially distributed student skills with comulative distribution $F(\alpha;\theta) = 1 - e^{-\frac{\alpha}{\theta}}$ and expected value $E[\alpha] = \theta$=1. For simplicity, we consider an example with N=20 and M=10.

Intuitively, low-skilled students and low-quality colleges would prefer the equilibrium with no signaling, since they could be matched with better partners than in a setting with costly signaling where the equilibrium match is assortative. Furthermore, in an environment with costly signaling, students must pay a cost that could make it even more difficult for them to have equilibrium payoffs above those of the game with no signaling, even for the highly qualified students.

According to the previous argument, agents may experiment gains or losses under the CSM with costly signaling with respect to the same setting with no signaling. We define the gains of implementing costly signaling under the CSM in a natural way, as the difference in agents' equilibrium payoffs between the separating signaling equilibrium and the symmetric equilibrium with no signaling. According to this definition, students' gains are defined as

$$L(\alpha) = \pi(\alpha, \rho_M(\alpha)) - EU(N, M) \tag{23}$$

Since the students' payoffs in the game with no signaling $EU(N, M)$ are type-independent, students' gains are strictly increasing in α. In addition, we know that the equilibrium payoffs satisfy $\pi(0, \rho_M(0)) = 0$ and $EU(N, M) > 0$, hence there always exit students who incur in losses under this comparison of equilibria of the CSM. Note that only the high-skilled students are willing to get positive gains depending on the prior distribution of skills.

In a similar way, college c_j's gains are defined as

$$\Delta EQ\left(j,N\right) = EQ^{*}_{c_j} - EQ_{c_j} \text{ for } j = 1,...,M. \tag{24}$$

Where $EQ^{*}_{c_j} = E\left[\alpha_{(j)}\right]$ is c_j's payoff in the separating equilibrium of the CSM and $EQ_{c_j} = \left(\frac{N-1}{N}\right)^{j-1} E\left[\alpha\right]$ is c_j's equilibrium payoff in the game with no signaling. As in the previous case, the behavior of colleges' gains depends on the prior distribution of student skills and several parameter of the model, such as the number of students and the number of colleges and school seats.

Under both kinds of equilibria of the CSM, a greater number of students leads colleges to increase equilibrium payoffs. In the case of the equilibrium with no signaling, payoffs increase because colleges reduce their probability of remaining unmatched. With costly signaling, an increase in the number of students improve the probability of enrolling better students, which translates into an improvement in colleges' payoffs. On the other hand, it is not clear which colleges get the highest gains or if colleges' gains are monotone in college quality.

In order to analyze colleges' gains in the CSM, it is necessary to characterize the moments of order statistics of the prior distribution of student skills. This is a difficult task because most distributions have no closed-form solutions for moments of order statistics. We focus on a particular prior distribution, the exponential distribution, for which it is possible to find closed-form formulas for moments of order statistics.[16] We consider the exponential model,[17] where $E\left[\alpha_{(j)}\right] = \sum_{k=1}^{N+1-j} \frac{\theta}{N+1-k}$ for $j = 1,...,M$ and $E\left[\alpha\right] = \theta$ (Huang, 1974). Under this

[16]The same results and conclusions can be obtained with the uniform distribution, for which it is also possible to derive closed-form solutions for moments of order statistics.

[17]Skills are exponentially distributed with parameter $\theta > 0$, if α is distributed according the density function $f\left(\alpha;\theta\right) = \frac{1}{\theta}e^{-\frac{\alpha}{\theta}}$. In this case, the cumulative distribution function is $F\left(\alpha;\theta\right) = 1 - e^{-\frac{\alpha}{\theta}}$. In addition, $E\left[\alpha\right] = \theta$ (Casella and Berger, 2002).

model, colleges' gains $\Delta EQ(j, N)$ satisfy the following expression

$$\Delta EQ(j, N) = \theta \sum_{k=1}^{N+1-j} \frac{1}{N+1-k} - \theta \left(\frac{N-1}{N} \right)^{j-1} \tag{25}$$

Then, we establish the following result.

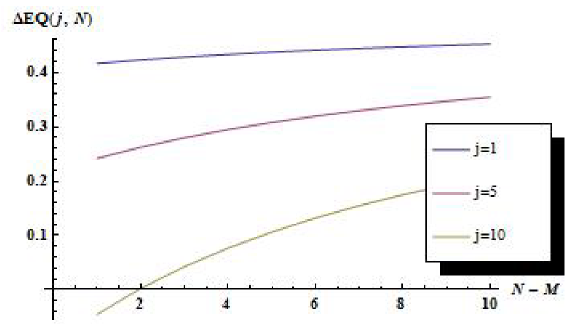
Proposition 7 *Consider any M by N college admissions problem such that $N > M \geq 1$. Assume that student skills are exponentially distributed with parameter $\theta > 0$. Then the following holds:*

1. *$\Delta EQ(j, N)$ is strictly monotonically increasing in N, i.e. $\Delta EQ(j, N+1) > \Delta EQ(j, N)$ for all $N > M$ and $j = 1, ..., M$;*
2. *$\Delta EQ(j, N)$ is strictly monotonically decreasing in j, i.e. $\Delta EQ(j+1, N) < \Delta EQ(j, N)$ for all $N > M$ and $j = 1, ..., M-1$; and*
3. *For any $M \geq 1$ there always exists an $N^* > M$ such that $\Delta EQ(j, N) \geq 0$ for all $j = 1, ..., M$ and all $N \geq N^*$.*

Proof. See Appendix C. ■

The previous result has interesting implications. First, an increasing demand for school seats improves the gains of colleges. Intuitively, an increasing pool of students leads colleges to reduce the risk of remaining unmatched and a costly signal becomes more effective to select the best available students from the pool of applicants. Another interesting implication regards the comparison of gains among colleges. As in the case of students, colleges' gains can be ranked according to college quality. This result implies that the big winners of the CSM are the high-quality colleges, which not only enroll the best students, but also earn the greatest gains from using a costly signal in college admission. The third interesting implication regards the relationship between the size of the demand for school seats and colleges' gains. We find that a big enough demand for school seats leads all colleges to earn positive gains. This result contrasts with the student case where there is always a proportion of students who incur in losses.

Figure 6: Colleges' gains with exponential distributed skills



Source: Based on equilibrium signaling strategy simulations of the CSM for exponentially distributed student skills with comulative distribution $F(\alpha;\theta) = 1 - e^{-\frac{\alpha}{\theta}}$ and expected value $E[\alpha] = \theta$=1. For simplicity, we consider an example with N=20 and M=10.

It is easy to show that the previous result cannot be trivially extended to any prior distribution of student skills. As we show in the following figure, we cannot guarantee neither the monotonicity of colleges' gains with respect to the number of students nor the monotonicity with respect to college quality. In this case, we consider Beta distributed skills with parameters $a = 10$ and $b = 2$.[18] Note that this distribution is skewed to the right. This fact may explain why the previous results about colleges' gains do not hold any more, since the probability of enrolling a good student with no signaling is significantly bigger.

7 Conclusion

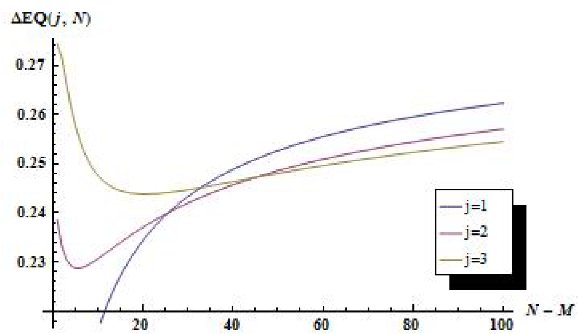
We analyze the role of costly signaling in a decentralized college admissions problem with incomplete information. We consider a matching problem where colleges with observable quality want to enroll students whose abilities are private information. We analyze a simple

[18]A random variable X is said to follow the **Beta distribution** with parameters (α, β) for some $\alpha > 0$ and $\beta > 0$, i.e. $X \sim Beta(\alpha, \beta)$, if Range(X) = (0,1) and for $x \in (0, 1)$ the density function satisfies:

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

Where $\Gamma(x)$ is the Gamma function.

Figure 7: Colleges' gains with $Beta(\alpha, \beta)$ distributed skills



Source: Based on equilibrium signaling strategy simulations of the CSM for beta distributed student skills with parameters $\alpha = 10$ and $\beta = 2$. For simplicity, we consider an example with N=110 and M=10.

decentralized matching mechanism called **Costly Signaling Mechanism** (CSM). Under the CSM, students choose a costly observable score to signal their skills. We characterize a pooling equilibrium with no signaling and a strictly separating symmetric equilibrium with costly signaling of the game induced by the CSM. Under the separating equilibrium of this game, the number of potential matches is maximized and agents are induced to efficient matches, in the sense that the best students will enroll in the best colleges.

We present three exercises of comparative statics that allow for analyzing the impact of a change in the underlying parameters of the model on the equilibrium signaling strategy. Our main results show that the effects are not symmetric across students, since they depend on student abilities. The first comparative statics exercise focuses on the effect of a change in the number of students. In this case, we show that an increase in the number of students leads low-skilled students to decrease their investment in signaling while the high-skilled students increase it. We also show that the effect of a change in the number of school seats and a change in college quality have similar implications.

Finally, we analyze the gains of the CSM, which are defined in a natural way as the difference in equilibrium payoffs between the separating signaling equilibrium and the pooling symmetric equilibrium of the game with no signaling. Under this definition, students' gains are strictly increasing with respect to the student skills, but, eventually, students may incur in losses depending on the prior distribution of skills. Since colleges' gains require the analysis of order statistics, we consider the particular case of exponentially distributed skills, which have a closed-form solution for colleges' gains. The exponential model has very interesting implications. First, colleges' gains are monotonically increasing in college quality. Second, colleges' gains are monotonically increasing in the number of students, i.e. all colleges benefit from an increasing demand for school seats. Finally, we show that a sufficiently large demand for school seats leads all colleges to earn positive gains.

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A Appendix

A.1 Appendix A: The Signaling Equilibrium

Proof of Proposition 1

Under the conditions of Proposition 1, it is clear that the condition $E[\alpha \mid P'] > E[\alpha \mid P]$ is satisfied for all $\alpha \in (0, w)$ whenever $P' > P$ where $\int \alpha g(\alpha \mid P)\, d\alpha = E[\alpha \mid P]$. Thus, higher scored students are associated with higher expected skills.

The payoffs of colleges are given by the expected quality of enrollees that depends on the outcome of the CSM (i.e. a matching between colleges and students) and the profile of student scores. So, given a matching μ, a college $c \in C$ has expected payoffs $E\left[\alpha \mid P_{\mu(c)}\right]$. A student $s \in S$ with parameter α gets a payoff of $v_c - C(\alpha, P_s)$ such that $\mu(s) = c$, and a payoff of $-C(\alpha, P_s)$ otherwise.

In order to simplify, we consider that after observing the profile of student scores $(P_s)_{s\in S}$, colleges "update" their ordinal preferences in the following simple way. Each college $c \in C$ forms an auxiliary preference relation $\succ_c^*$ over the set of students and the prospect of remaining unmatched, $S \cup \{c\}$ such that (i) $s \succ_c^* c$ if and only if $P_s > 0$ and (ii) for any $s, s' \in S$, $s \succ_c^* s'$ if and only if $P_s > P_{s'}$. Note that the profile of interim preferences$\succ_C^* = \left(\succ_c^*\right)_{c\in C}$ is consistent with beliefs $G(\alpha \mid P)$.

In the matching stage of the CSM, the profile of student scores is given, so colleges form the interim preference $\succ_C^* = \left(\succ_c^*\right)_{c\in C}$ while the signaling cost is a sunk cost for students. This implies that student preferences in this stage are well defined and coincide with the strict preferences order $\succ_S = (\succ_s)_{s\in S}$. Assume w.l.g. that colleges have different quality, i.e $v_1 > v_2 > .. > v_M > 0$. Also, assume that students' scores satisfy $P_s \neq P_{s'}$ for all $s, s' \in S$. This assumption about student scores is not strong, since in equilibrium ties will happen with probability zero. Then, for any given profile of student scores $(P_s)_{s\in S}$, there is a well defined college admissions problem in the matching stage of the CSM with strict preferences denoted by $\left(S, C, \left(\succ_S, \succ_C^*\right)\right)$.

Note that the matching problem $\left(S, C, \left(\succ_S, \succ_C^*\right)\right)$satisfies the conditions of the theorem by Alcalde and Romero-Medina (1998) in the matching stage of the CSM, hence only stable matchings between students and colleges are a reasonable SPE outcomes of the CSM. Furthermore, given that colleges and students have the same preferences, this problem has a unique stable matching that is assortative. This completes the proof.

The maximization problem of any student with parameter α is:

$$\max_{P_1 \geq 0} \left\{ \sum_{k=1}^{M} v_k \binom{N-1}{k-1} F\left(\rho_N^{-1}(P_1)\right)^{N-k} \left[1 - F\left(\rho_N^{-1}(P_1)\right)\right]^{k-1} - \frac{c(P_1)}{\phi(\alpha)} \right\} \tag{26}$$

Defining the function $\varphi(x, N, k) = F(x)^{N-k}[1 - F(x)]^{k-1}$, for each $k \in \{2, ..., N-1\}$ we have

$$\varphi'(x, N, k) = [(N-k)(1 - F(x)) - (k-1) F(x)] F(x)^{N-1-k} [1 - F(x)]^{k-2} f(x) \tag{27}$$

Hence, the FOC of the payoff function $\pi(\alpha, P_1)$ with respect to P_1 is given by:

$$\begin{gathered} v_1 (N-1) F\left(\rho_N^{-1}(P_1)\right)^{N-2} \frac{f\left(\rho_N^{-1}(P_1)\right)}{\rho_N'\left(\rho_N^{-1}(P_1)\right)} + \\ + \sum_{k=2}^{M} v_k (N-k) \tbinom{N-1}{k-1} F\left(\rho_N^{-1}(P_1)\right)^{N-1-k} \left[1 - F\left(\rho_N^{-1}(P_1)\right)\right]^{k-1} \frac{f\left(\rho_N^{-1}(P_1)\right)}{\rho_N'\left(\rho_N^{-1}(P_1)\right)} - \\ - \sum_{k=2}^{M} v_k (k-1) \tbinom{N-1}{k-1} F\left(\rho_N^{-1}(P_1)\right)^{N-k} \left[1 - F\left(\rho_N^{-1}(P_1)\right)\right]^{k-2} \frac{f\left(\rho_N^{-1}(P_1)\right)}{\rho_N'\left(\rho_N^{-1}(P_1)\right)} - \\ - \frac{c'(P_1)}{\phi(\alpha)} = 0 \end{gathered} \tag{28}$$

In a symmetric equilibrium $P_1 = \rho_M(\alpha)$, then

$$\begin{gathered} v_1 (N-1) \phi(\alpha) F(\alpha)^{N-2} f(\alpha) + \\ + \sum_{k=2}^{M} v_k (N-k) \tbinom{N-1}{k-1} \phi(\alpha) F(\alpha)^{N-1-k} [1 - F(\alpha)]^{k-1} f(\alpha) - \\ - \sum_{k=2}^{M} v_k (k-1) \tbinom{N-1}{k-1} \phi(\alpha) F(\alpha)^{N-k} [1 - F(\alpha)]^{k-2} f(\alpha) = \\ = c'(\rho_N(\alpha)) \rho_N'(\alpha) \end{gathered} \tag{29}$$

By reordering and solving this differential equation with the initial condition $\rho_M(0) = 0$, we find that the signaling strategy $\rho_M(\alpha)$ satisfies

$$\rho_M(\alpha) = \tag{30}$$
$$= c^{-1}\left(\begin{array}{c} (N-1)\sum_{k=1}^{M-1}(v_k - v_{k+1})\binom{N-2}{k-1}\int_0^{\alpha}\phi(x)F(x)^{N-1-k}[1-F(x)]^{k-1}f(x)\,dx + \ldots \\ \ldots + (N-1)v_M\binom{N-2}{M-1}\int_0^{\alpha}\phi(x)F(x)^{N-M-1}[1-F(x)]^{M-1}f(x)\,dx \end{array}\right)$$

This fully characterizes the maximization problem of any student with parameter α. Note that $(N-1)\binom{N-2}{k-1} = \frac{(N-1)!}{(k-1)!(N-1-k)!}$, so we can re-write this signaling strategy as

$$\rho_M(\alpha) = \tag{31}$$
$$= c^{-1}\left(\sum_{k=1}^{M-1}(v_k - v_{k+1})\int_0^{\alpha}\phi(x)f_{(k,N-1)}(x)\,dx + v_M\int_0^{\alpha}\phi(x)f_{(M,N-1)}(x)\,dx\right)$$

Where $f_{(k,N-1)}(x) = \frac{(N-1)!}{(k-1)!(N-1-k)!}F(x)^{N-1-k}[1-F(x)]^{k-1}f(x)$ for $k = 1,\ldots,M$. with $f_{(k,N-1)}(x)$ being the density probability function of the x-th highest order statistic from an iid sample of size $N-1$.

Proof of Proposition 2

Consider that any student with parameter α is planning to deviate from the signaling strategy $\rho_M(\alpha)$ by choosing an alternative score P'. Assume w.l.g. that $0 \leq P' < \rho_M(\alpha)$, since the signaling strategy is strictly increasing in α, there exists a unique $0 \leq \alpha' < \alpha$ such that $P' = \rho_M(\alpha')$. Then, by choosing the score P' the student gets the expected payoff $\pi(\alpha, P') = \pi(\alpha, \rho_M(\alpha'))$ given by

$$\pi(\alpha, \rho_M(\alpha')) = \sum_{k=1}^{M} v_k\binom{N-1}{k-1}F(\alpha')^{N-k}[1-F(\alpha')]^{k-1} - \frac{c(\rho_M(\alpha'))}{\phi(\alpha)} \tag{32}$$

By deviating to P', the student loses the extra payoff of

$$\pi\left(\alpha,\rho_M\left(\alpha\right)\right)-\pi\left(\alpha,\rho_M\left(\alpha'\right)\right)=$$
$$=\sum_{k=1}^{M}v_k\binom{N-1}{k-1}\left(F\left(\alpha\right)^{N-k}\left[1-F\left(\alpha\right)\right]^{k-1}-F\left(\alpha'\right)^{N-k}\left[1-F\left(\alpha'\right)\right]^{k-1}\right)- \tag{33}$$
$$-\frac{1}{\phi(\alpha)}\left(c\left(\rho_M\left(\alpha\right)\right)-c\left(\rho_M\left(\alpha'\right)\right)\right)$$

The increment in the signaling cost $c\left(\rho_M\left(\alpha\right)\right)-c\left(\rho_M\left(\alpha'\right)\right)$ is positive and equal to

$$c\left(\rho_M\left(\alpha\right)\right)-c\left(\rho_M\left(\alpha'\right)\right)=$$
$$=\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\int_{\alpha'}^{\alpha}\phi\left(x\right)f_{(k,N-1)}\left(x\right)dx+v_M\int_{\alpha'}^{\alpha}\phi\left(x\right)f_{(M,N-1)}\left(x\right)dx \tag{34}$$

Since $\phi\left(x\right)$ is strictly increasing and positive in x, it is clear that

$$\frac{1}{\phi(\alpha)}\left(c\left(\rho_M\left(\alpha\right)\right)-c\left(\rho_M\left(\alpha'\right)\right)\right)<$$
$$<\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\int_{\alpha'}^{\alpha}f_{(k,N-1)}\left(x\right)dx+v_M\int_{\alpha'}^{\alpha}f_{(M,N-1)}\left(x\right)dx \tag{35}$$

By reordering the previous equation, we find that the following condition holds

$$\frac{1}{\phi(\alpha)}\left(c\left(\rho_M\left(\alpha\right)\right)-c\left(\rho_M\left(\alpha'\right)\right)\right)<$$
$$<\left(N-1\right)v_1\int_0^{\alpha}F\left(x\right)^{N-2}f\left(x\right)dx+\sum_{k=2}^{M}v_k\binom{N-1}{k-1}\int_0^{\alpha}\varphi'\left(x,N,k\right)dx \tag{36}$$

For any $k\in\{2,...,N-1\}$, we have that

$$\int_{\alpha'}^{\alpha}\varphi'\left(x,N,k\right)dx=F\left(\alpha\right)^{N-k}\left[1-F\left(\alpha\right)\right]^{k-1}-F\left(\alpha'\right)^{N-k}\left[1-F\left(\alpha'\right)\right]^{k-1} \tag{37}$$

which implies that $\pi\left(\alpha,\rho_M\left(\alpha\right)\right)-\pi\left(\alpha,\rho_M\left(\alpha'\right)\right)>0$. By a similar argument, it is possible to show that any deviation $P'>\rho_N\left(\cdot\right)$ cannot be a profitable deviation. This completes the proof.

Proof of Proposition 3

- $\rho_M\left(\alpha\right)$ is strictly increasing in α and bounded from above.

To prove that the signaling strategy $\rho_M\left(\alpha\right)$ is strictly increasing in α, it is enough to show that the function $c(\rho_M(\alpha))$ is strictly increasing in α, since $c(\cdot)$ is a strictly increasing function. Then,

$$\begin{gathered}\frac{d}{d\alpha}\left(c\left(\rho_M(\alpha)\right)\right)= \\ =(N-1)\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\binom{N-2}{k-1}\phi(\alpha)F(\alpha)^{N-1-k}\left[1-F(\alpha)\right]^{k-1}f(\alpha)+\ldots \\ \ldots+(N-1)v_M\binom{N-2}{M-1}\phi(\alpha)F(\alpha)^{N-M-1}\left[1-F(\alpha)\right]^{M-1}f(\alpha)\end{gathered} \tag{38}$$

It is clear that $\frac{d}{d\alpha}\left(c\left(\rho_M(\alpha)\right)\right)>0$ for all α, as we desired. To prove that the signaling strategy $\rho_M(\alpha)$ is bounded from above, we use the fact that this function can be written as

$$\begin{gathered}c\left(\rho_M(\alpha)\right)= \\ =\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\int_0^{\alpha}\phi(x)f_{(k,N-1)}(x)\,dx+v_M\int_0^{\alpha}\phi(x)f_{(M,N-1)}(x)\,dx\end{gathered} \tag{39}$$

where $f_{(k,N-1)}(x)$ is the density function of the $k-th$ order statistic from an $N-1$ sample with distribution function $F(x)$, such that $x_{(1,N-1)}=\underset{1\leq i\leq N-1}{max}\{x_i\}$, $x_{(2,N-1)}$=second greatest in $\{x_i\}_{i=1}^{N-1}$ and so on. Since $\phi(x)$ is strictly increasing and bounded in $[0,w]$, we know that

$$c\left(\rho_M(\alpha)\right)\leq\phi(w)\left(\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\int_0^{w}f_{(k,N-1)}(x)\,dx+v_M\int_0^{w}f_{(M,N-1)}(x)\,dx\right) \tag{40}$$

But by definition $\int_0^{w}f_{(k,N-1)}(x)\,dx=1$ for all $k=1,\ldots,M$. Then,

$$c\left(\rho_M(\alpha)\right)\leq\phi(w)\left(\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)+v_M\right)<\infty \tag{41}$$

- $\pi(\alpha,\rho_M(\alpha))$ is strictly increasing in α .

Now we want to show that the equilibrium payoff $\pi(\alpha,\rho_M(\alpha))$ is strictly increasing in α. To prove this property, we calculate the derivative of the payoff function with respect to α

$$\frac{d}{d\alpha}\left(\pi\left(\alpha,\rho_M\left(\alpha\right)\right)\right)=\begin{cases} \left(N-1\right)v_1F\left(\alpha\right)^{N-2}f\left(\alpha\right)+\sum\limits_{k=2}^{M}v_k\binom{N-1}{k-1}\varphi'\left(\alpha\right)- \\ -\frac{1}{\phi(\alpha)^2}\left(\phi\left(\alpha\right)\frac{d}{d\alpha}\left(c\left(\rho_M\left(\alpha\right)\right)\right)-c\left(\rho_M\left(\alpha\right)\right)\phi'\left(\alpha\right)\right)\end{cases} \tag{42}$$

By reordering the previous expression, it is easy to show that

$$\frac{d}{d\alpha}\left(\pi\left(\alpha,\rho_M\left(\alpha\right)\right)\right)=\frac{c\left(\rho_M\left(\alpha\right)\right)\phi'\left(\alpha\right)}{\phi\left(\alpha\right)^2}>0 \tag{43}$$

This completes the proof.

A.2 Appendix B: Comparative Statics

Proof of Proposition 4

Consider the equilibrium signaling strategy of any college admissions problem with one college with $M\geq 1$ school seats and $N>M$ students.

$$\begin{gathered}\rho_M\left(\alpha,N\right)= \\ =c^{-1}\left(\ \left(N-1\right)v_1\binom{N-2}{M-1}\int_0^{\alpha}\phi\left(x\right)F\left(x\right)^{N-1-M}\left[1-F\left(x\right)\right]^{M-1}f\left(x\right)dx\ \right)\end{gathered} \tag{44}$$

Since the cost function $c\left(\cdot\right)$ is strictly increasing, it is enough to show that the function $c\left(\rho_M\left(\alpha,N\right)\right)$ satisfies the desired properties. The difference $c\left(\rho_M\left(\alpha,N+1\right)\right)-c\left(\rho_M\left(\alpha,N\right)\right)$ is equal to

$$\begin{gathered}v_1\int_0^{\alpha}\left(N\binom{N-1}{M-1}F\left(x\right)-\left(N-1\right)\binom{N-2}{M-1}\right)\times \\ \times\phi\left(x\right)F\left(x\right)^{N-1-M}\left[1-F\left(x\right)\right]^{M-1}f\left(x\right)dx\end{gathered} \tag{45}$$

Then, it is clear that $c\left(\rho_M\left(\alpha,N+1\right)\right)-c\left(\rho_M\left(\alpha,N\right)\right)<(>)0$ if $\alpha<(>)\alpha_N\left(N\right)$, where $\alpha_N\left(N\right)$ is the unique solution to the equation

$$F\left(x\right)=1-\frac{M}{N} \tag{46}$$

Further, the threshold $\alpha_N\left(N\right)$ is monotonically increasing in N, i.e. $\alpha_N\left(N+1\right)>\alpha_N\left(N\right)$

for all $N > M$. This completes the proof.

Proof of Proposition 5

Consider the equilibrium signaling strategy of any college admissions problem with one college with $M \geq 1$ school seats and $N > M$ students.

$$\begin{gathered}\rho_M(\alpha, M) = \\ = c^{-1}\left((N-1)\, v_1 \binom{N-2}{M-1} \int_0^{\alpha} \phi(x)\, F(x)^{N-1-M} [1-F(x)]^{M-1} f(x)\, dx \right)\end{gathered} \tag{47}$$

Since the function $c(\cdot)$ is strictly increasing, to prove this result we focus on the function $c(\rho_M(\alpha, M))$. The difference $c(\rho_M(\alpha, M+1)) - c(\rho_M(\alpha, M))$ is equal to

$$\begin{gathered}v_1 (N-1) \int_0^{\alpha} \left(\binom{N-2}{M} (1-F(x)) - \binom{N-2}{M-1} F(x) \right) \times \\ \times \phi(x)\, F(x)^{N-2-M} [1-F(x)]^{M-1} f(x)\, dx\end{gathered} \tag{48}$$

By reordering and applying the identity $\binom{N}{k} = \binom{N-1}{k-1} + \binom{N-1}{k}$, the previous equation becomes

$$\begin{gathered}v_1 (N-1) \int_0^{\alpha} \left(\binom{N-2}{M} - \binom{N-1}{M} F(x) \right) \times \\ \times \phi(x)\, F(x)^{N-2-M} [1-F(x)]^{M-1} f(x)\, dx\end{gathered} \tag{49}$$

Given that $\binom{N-1}{M} = \frac{N-1}{N-1-M} \binom{N-2}{M}$, we have that

$$\begin{gathered}v_1 (N-1) \binom{N-2}{M} \int_0^{\alpha} \left(1 - \frac{N-1}{N-1-M} F(x) \right) \times \\ \times \phi(x)\, F(x)^{N-2-M} [1-F(x)]^{M-1} f(x)\, dx\end{gathered} \tag{50}$$

Then, it is clear that $c(\rho_M(\alpha, M+1)) - c(\rho_M(\alpha, M)) > 0$ if $\alpha \leq \alpha_N(M, N)$, where $\alpha_M(M, N)$ is the unique solution to the equation

$$F(x) = 1 - \frac{N-1-M}{N-1} \tag{51}$$

Further, the threshold $\alpha_M(M, N)$ is monotonically increasing in N and monotonically decreasing in M. This completes the proof.

Proof of Proposition 6

We analyze the effect of a change in the quality of the college c_k considering that the equilibrium signaling strategy depends on the quality of this college, i.e. $\rho_M(\alpha, v_k)$. We know that the equilibrium signaling strategy satisfies the equation

$$c(\rho_M(\alpha, v_k)) = \sum_{k=1}^{M} v_k \binom{N-1}{k-1} \int_0^{\alpha} \phi(x)\, \varphi'(x, N, k)\, dx \tag{52}$$

Consider a change in the quality of the college c_k such that $v_{k-1} > v_k' > v_{k+1}$, i.e. this change in college quality preserves students' ordinal preferences. The difference $c\left(\rho_M\left(\alpha, v_k'\right)\right) - c(\rho_M(\alpha, v_k))$ satisfies the following equation for $k = 1, ..., M$

$$c\left(\rho_M\left(\alpha, v_k'\right)\right) - c(\rho_M(\alpha, v_k)) = \left(v_k' - v_k\right) \binom{N-1}{k-1} \int_0^{\alpha} \phi(x)\, \varphi'(x, N, k)\, dx \tag{53}$$

Since $\varphi'(x, N, k) = [(N-k)(1-F(x)) - (k-1)F(x)]\, F(x)^{N-1-k} [1-F(x)]^{k-2} f(x)$, it is easy to show that whenever $\alpha \leq \alpha_{v_k}(N, k)$

$$sgn\left(c\left(\rho_M\left(\alpha, v_k'\right)\right) - c(\rho_M(\alpha, v_k))\right) = sgn\left(v_k' - v_k\right) \tag{54}$$

Where $\alpha_{v_k}(N, k)$ is the unique solution to the equation

$$F(x) = \frac{N-k}{N-1} \tag{55}$$

Further, observe that the threshold $\alpha_{v_k}(N, k)$ is monotonically increasing in N for all $k = 2, ..., M$ and monotonically decreasing in k. This completes the proof.

A.3 Appendix C: Gains of the CSM

If α is distributed according to an exponential distribution function $f(\alpha; \theta) = \frac{1}{\theta} e^{-\frac{\alpha}{\theta}}$ for $\alpha \in [0, \infty)$ and $\theta > 0$, then

1. $E[\alpha] = \theta$ and

2. $E\left[\alpha_{(j)}\right] = \sum_{k=1}^{N+1-j} \frac{\theta}{N+1-k}$ for $j = 1, ..., N$.

where $\alpha_{(1)} = \max_{1\leq i\leq N} \alpha_i$, $\alpha_{(2)}$=second greatest in $\{\alpha_i\}_{i=1}^{N-1}$and so on (Huang, 1974). Consider any M by N college admissions problem such that $N > M \geq 1$, then colleges' gains satisfy

$$\Delta EQ(j, N) = \theta \sum_{k=1}^{N+1-j} \frac{1}{N+1-k} - \theta \left(\frac{N-1}{N}\right)^{j-1} \tag{56}$$

Assume w.l.g. that $\theta = 1$. We establish the following auxiliary results.

Claim 1 *The continuous function $f(x) = \left(\frac{x-1}{x}\right)^{j-1}$ is strictly increasing and strictly concave in x for all $x > j \geq 3$.*

Proof. To prove this result, we simply take the first and second derivatives of the function $f(x) = \left(\frac{x-1}{x}\right)^{j-1}$ and obtain that

1. $f'(x) = \left(\frac{j-1}{x^2}\right)\left(\frac{x-1}{x}\right)^{j-2} > 0$; and

2. $f''(x) = \left(\frac{j-1}{x^4}\right)\left(\frac{x-1}{x}\right)^{j-3}(j-2x) < 0$.

For all $x > j \geq 3$, this completes the proof. ■

Lemma 1 $\Delta EQ(j, N) > \Delta EQ(j+1, N)$ *for all* $j = 1, ..., M-1$.

Proof. The difference $\Delta EQ(j, N) - \Delta EQ(j+1, N)$ satisfies

$$\Delta EQ(j, N) - \Delta EQ(j+1, N) = \frac{1}{j} - \left(\frac{N-1}{N}\right)^{j-1}\left(\frac{1}{N}\right) \tag{57}$$

Since $\left(\frac{N-1}{N}\right)^{j-1} \leq 1$ for all $j \geq 1$, we know that

$$\Delta EQ(j, N) - \Delta EQ(j+1, N) \geq \frac{1}{j} - \frac{1}{N} = \frac{N-j}{jN} \tag{58}$$

Given that $N > M \geq j$, $\Delta EQ(j, N) - \Delta EQ(j+1, N) > 0$ for $j = 1, ..., M-1$. This completes the proof. ■

Lemma 2 *$\Delta EQ\,(j, N)$ is strictly monotone increasing in $N > M$ for all $j = 1, ..., M$.*

Proof. Consider the following function for a given $j = 1, ..., M$

$$\Delta EQ\,(j, N+1) - \Delta EQ\,(j, N) = \begin{cases} \sum_{k=1}^{N+2-j} \frac{1}{N+2-k} - \sum_{k=1}^{N+1-j} \frac{1}{N+1-k} - \\ - \left[\left(\frac{N}{N+1}\right)^{j-1} - \left(\frac{N-1}{N}\right)^{j-1} \right] \end{cases} \tag{59}$$

By simplifying, we obtain that

$$\Delta EQ\,(j, N+1) - \Delta EQ\,(j, N) = \frac{1}{N+1} - \left(\left(\frac{N}{N+1}\right)^{j-1} - \left(\frac{N-1}{N}\right)^{j-1} \right) \tag{60}$$

By a direct inspection we see that $\Delta EQ\,(j, N+1) - \Delta EQ\,(j, N) > 0$ for $j = 1, 2$. Now consider the case of any j such that $N > M \geq j \geq 3$. By Claim 1, we know that

$$f'(N) \geq \left(\frac{N}{N+1}\right)^{j-1} - \left(\frac{N-1}{N}\right)^{j-1} \tag{61}$$

where $f\,(x) = \left(\frac{x-1}{x}\right)^{j-1}$ such that $x > j \geq 3$. Hence,

$$\Delta EQ\,(j, N+1) - \Delta EQ\,(j, N) \geq \frac{1}{N+1} - \left(\frac{j-1}{N^2}\right)\left(\frac{N-1}{N}\right)^{j-2} \tag{62}$$

Since $\left(\frac{N-1}{N}\right)^{j-2} \leq 1$ for all $j \geq 2$, we know that

$$\Delta EQ\,(j, N+1) - \Delta EQ\,(j, N) \geq \frac{1}{N+1} - \frac{j-1}{N^2} = \frac{N^2 - (j-1)(N+1)}{N^2\,(N+1)} \tag{63}$$

Given that $N > M \geq j \geq 3$ and $(N-1)(N+1) > (j-1)(N+1)$, we conclude that

$$\Delta EQ\,(j, N+1) - \Delta EQ\,(j, N) > \frac{1}{N^2\,(N+1)} \tag{64}$$

Then, $\Delta EQ\,(j, N+1) - \Delta EQ\,(j, N) > 0$ for all $N > M \geq j \geq 3$. This completes the proof. ■

Proof of Proposition 7

Properties 1 and 2 of colleges' gains $\Delta EQ\,(j,N)$ come directly from Lemmas 1 and 2. For the third property, assume that $\Delta EQ\,(M,N) \geq 0$ for $N = M+1$, then $N^* = M+1$. By Lemma 2, $\Delta EQ\,(M,N) \geq 0$ for all $N \geq N^* > M$. By Lemma 1, $\Delta EQ\,(j,N) \geq 0$ for all $j = 1,\ldots,M$, provided that $\Delta EQ\,(M,N) \geq 0$. Then, $\Delta EQ\,(j,N) \geq 0$ for all $N \geq N^*$ and $j = 1,\ldots,M$.

Now suppose that $\Delta EQ\,(M,N) < 0$ for $N = M+1$. Note that

1. $\lim\limits_{N\to\infty} \left(\frac{N-1}{N}\right)^{M-1} = 1$ for all $M \geq 1$; and

2. $\lim\limits_{N\to\infty} E\left[\alpha_{(M)}\right] = \lim\limits_{N\to\infty} \sum\limits_{k=1}^{N+1-M} \frac{1}{N+1-k} = \infty.$

Then, there exists a $N^* > M$ such that $\Delta EQ\,(M,N^*) \geq 0$. By Lemmas 1 and 2, $\Delta EQ\,(j,N) \geq 0$ for all $N \geq N^*$ and $j = 1,\ldots,M$. This completes the proof.