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The Equilibrium Set of Economies with a Continuous Consumption Space*

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Abstract

We study global properties of the equilibrium set of economies with a continuous consumption space. This framework is important in intertemporal allocation problems (continuous or infinite time), financial markets with uncertainty (continuous states of nature) and commodity differentiation. We show that the equilibrium set is contractible which implies that (i) there is a continuous economic policy linking any two equilibrium states, and (ii) any two such economic policies can be continuously deformed one into the other. We also give three equivalent formulations of the problem of global uniqueness of equilibria in terms of the projection map from the equilibrium set to the space of parameters. We finally study the local and global effects that the existence of critical economies has on the equilibrium set.
Keywords: General equilibrium; infinite economies; intertemporal choice; uncertainty.
JEL Classification: D50, D51, D80, D90.

Resumen

Estudiamos propiedades globales del conjunto de equilibrio para economías con un espacio de consumo continuo. Este marco es importante en problemas de asignación intertemporal (tiempo continuo o infinito), mercados financieros con incertidumbre (estados de la naturaleza continuos) y diferenciación de productos. Demostramos que el conjunto de equilibrio es contraíble lo que implica que (i) existe una política económica continua uniendo dos estados de equilibrio cualesquiera , y (ii) cualesquiera dos de estas políticas económicas pueden ser deformadas continuamente una en la otra. Asimismo, proponemos tres formulaciones equivalentes del problema de unicidad global de equilibrios en términos de la proyección que va del conjunto de equilibrio al espacio de parámetros. Finalmente, estudiamos los efectos locales y globales que la existencia de economías críticas genera en el conjunto de equilibrio.
Palabras Clave: Equilibrio general; economías infinitas; elección intertemporal; incertidumbre.

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1 Introduction

The study of economies with commodity spaces of infinite dimensions is at the cornerstone of modern economic theory. It is of genuine interest within microeconomic theory but it is also a framework that is needed to study problems of, for example, financial markets, growth theory, dynamic general equilibrium and commodity differentiation.

The theory of general equilibrium in finite dimensions is an area that has made much progress over the years but a direct extension of the theory to infinite economies has been challenging and results do not generalize in many cases. Even selecting the right commodity space is far from trivial since, for instance, consider that if a pure exchange economy has ℓ goods, the choice of a commodity space is a simple task since all ℓ-dimensional vector spaces are isomorphic to $\mathbb{R}^{\ell}$. In this case, the consumption space is a subset of the positive cone of $\mathbb{R}^{\ell}$. However, on the opposite end, there is not a single vector space to which all other infinite dimensional vector spaces are isomorphic to and hence, with an infinite number of goods, the choice of a commodity set may give rise to very different frameworks.

Representing preferences is also problematic. Recent work of Hervés-Beloso and Monteiro [17] shows that if we take preferences as primitives, unless we consider the space of continuous functions (or of integrable functions) as the commodity space it is impossible to represent strictly monotonic preferences by a continuous utility function. Negative results also appear when studying individual demand functions. A result of Araujo [3] showed that when the commodity space is a general Banach space[1] a demand function will exist if and only if the commodity space is reflexive. Also, even if the demand function exists, it will be continuously differentiable if and only if the commodity space is actually a Hilbert space.

Given these trade-offs, in our previous work ([10], [11]) we set to study models of pure exchange economies with a continuous commodity space. As we will exemplify below, this model is a good framework to study problems in finance, infinite-horizon models and product differentiation. We used this model in [10] to study the equilibrium set, to study regular and critical economies (and regular and critical prices) and as a by-product we gave a new proof the genericity of regular economies. We also studied the structure of excess demand functions and constructed an index theorem in [11] which, among other results, gave generic necessary and sufficient conditions for global uniqueness of equilibria.

This paper aims to characterise the equilibrium set and derive properties in several directions. We begin in section 2 by setting the model and in section 3 we will investigate some global properties of the equilibrium set analogous to the Balasko programme (see [5]) for finitely many goods. We show that the equilibrium set has the structure of a Banach manifold, and that it is a contractible set so that in particular it is arc-connected and simply connected.

[1] An appendix with mathematical definitions is included at the end of the paper.

The topological triviality of the equilibrium set gives analogous results to the finite-dimensional case: suppose that p is a price, ω the parameter that defines an economy, that (p, ω) describes a current equilibrium state and that (p', ω') is an exogenously determined new equilibrium that is to be achieved at a future. Every path connecting these two equilibria is then the mathematical expression of some economic policy[2]. Connectedness of the equilibrium set then proves that there is a continuous economic policy linking equilibria (p, ω) and (p', ω') while arc-connectedness expresses the idea that any two such economic policies can be continuously deformed one into the other.

As an application of the simplicity of the equilibrium set, in section 4 we give a purely topological characterisation of the problem of global uniqueness of equilibria in terms of the projection map and its relation with critical equilibria. In section 5 we give a brief review of the definitions and main results of Fredholm index theory and we show that the projection map is Fredholm of index zero. Finally, in section 6 we give three equivalent analytical formulations of the problem of global uniqueness of equilibria and study the local and global effects that the set of critical economies has on the shape of the equilibrium set. To aid comprehension, we add an appendix with mathematical definitions.

[2]That is, formally an *economic policy* is a continuous map $\mathcal{P}(\tau)$, $\tau \in [0,1]$ such that $\mathcal{P}(0) = (p, \omega)$, $\mathcal{P}(1) = (p', \omega')$ and such that at each $\tau \in (0,1)$, $\mathcal{P}(\tau)$ is an equilibrium.

2 The Market

We give three examples of economies with a continuous consumption space. Further references can be found in [20].

2.1 Examples of continuous economies

Example 1. Financial Markets. The following example is a particular case of [12] and [19] where we consider a two-time period $t = 0, 1$ economy with complete financial markets and uncertainty at the second time period. The set of states is $M = [0, 1]$ and the C^1 map $\pi : M \to \mathbb{R}_+$ is the density of the set of states M. We suppose there is a finite number $i = 1, \ldots, I$ of consumers. A consumption bundle is a pair $x_i = (x_i^0, x_i^1)$ where at $t = 0$ consumption is a vector $x_i^0 \in \mathbb{R}^\ell_{++}$ and at $t = 1$ it is a C^1 map $x_i^1 : M \to \mathbb{R}^\ell_+$. Similarly, a price is a pair $p = (p_0, p_1)$, where $p_0 \in \mathbb{R}^\ell_+$ and at $t = 1$, it is a map $p_1 : M \to \mathbb{R}^\ell_+$.

We suppose that agents are equipped with a $t = 0$ endowment $\omega_i^0 \in \mathbb{R}^n_{++}$ and a C^1 initial endowment at $t = 1$ of the form $\omega_i^1 : M \to \mathbb{R}^\ell_{++}$. Preferences are represented by a state-dependent utility of the form

$$U^i(x_i) = u_i(x_i^0) + \int_M u^i\left(x_i^1(s)\right) \pi(s)\, ds.$$

It is shown in [12] and [19] that if $(p, x_1, \ldots x_I)$ is an equilibrium, then p_1 and x_i^1 for each i are all continuous maps from M to $\mathbb{R}^\ell_{++}$. In other words, prices, consumption and endowments are all elements of the same space $C(M, \mathbb{R}^{\ell}_{++})$.

Example 2. Continuous time. Suppose that in an economy the consumption of n goods is done continuously through time $t \in [0, T]$. Then, a continuous function $x^i : [0, T] \to \mathbb{R}^n_{++}$ represents the consumption of the n goods by agent i at time t. Alternatively, $x(t)$ may represent a continuous instantaneous rate of consumption.

Example 3. Commodity differentiation. In this final example, we mention economies that allow product differentiation. In this case there is a compact (topological) set M of "characteristics" and a set $\mathcal{U} \subset M$ would represent a specific subset of characteristics that a commodity bundle is desired to have. For each set $\mathcal{U} \subset M$, a continuous function $x : \mathcal{U} \to \mathbb{R}_+$ represents the proportion of commodity bundles satisfying the characteristics given by $\mathcal{U}$. Naturally, we must have $\int_M x(t)\, dt = 1$. This is a continuous version of example (C) in ([20], p.1837).

2.2 The exchange economy

2.2.1 The commodity space

With the examples of the previous section in mind we assume that the commodity space is a subset of $C(M, \mathbb{R}^n)$, the set of continuous maps with the sup-norm topology, where M, the parameter space, is a compact subset of $\mathbb{R}^a$ for some a.[3] This commodity space has several additional advantages: (i)

[3] The assumption of M being compact is not as restrictive as it might initially seem. Similar in spirit to this paper are [2], [6], [7] and [8] that study the (non-compact) infinite-

the interior of its positive cone (the consumption space) is non-empty, (ii) it will allow us to write down in a natural way separability of utilities that will simplify our analysis, and (iii) the price space will have a particularly simple structure as we will explain below.

2.2.2 The consumption space

The **consumption space** is then $X = C^{++}(M, \mathbb{R}^n)$, the positive cone of $C(M, \mathbb{R}^n)$; that is, the subset of maps in $C(M, \mathbb{R}^n)$ for which its range consists solely of positive entries. In other words, the consumption plan of agent i is a continuous function $x_i : M \to \mathbb{R}^n_{++}$.

2.2.3 Initial endowments

We consider a finite number $i = 1, \ldots, I$ of agents each of which is equipped with initial endowments $\omega_i \in X$. In other words, the initial endowment of agent i is a continuous function $\omega_i : M \to \mathbb{R}^n_{++}$.

2.2.4 Preferences

For each agent i, preferences are represented by utilities of the form

$$U_i(x) = \int_M u^i(x(t), t)\, dt.$$

horizon model through the natural projection approach. A difference with those papers is that we do not parametrize economies by the welfare weights and instead we choose a direct parametrization by initial endowments.

We assume that the parameter-dependent utility function $u^i(x(t),t) : \mathbb{R}^n_{++} \times M \to \mathbb{R}$ is a strictly monotonic, concave, C^2 function where $\{y \in R^n_{++} : u^i(y,t) \geq u^i(x,t)\}$ is closed. This implies that $U_i(x)$ is strictly monotonic, concave and twice Fréchet differentiable.

The advantage of assuming separable utilities, is that we are decomposing an infinite-dimensional optimization problem into an infinite sequence but of finite dimensional problems.

2.2.5 Prices

Strictly speaking, prices are in the positive cone of the dual of $C(M, \mathbb{R}^n)$. However, Crés et al [12] and Chichilnisky and Zhou [9] have shown that an economy with continuous endowment and with separable utilities, prices must be continuous functions of the parameter space M and so we can then simply consider the **price space** to be

$$S = \left\{p \in C^{++}(M, \mathbb{R}^n) : \|p\| = 1\right\},$$

where

$$\|p\| = \sup_{t \in M} \|p(t)\|$$

with the standard metric $\|\cdot\|$ on $\mathbb{R}^n$. We denote by $\langle \cdot, \cdot \rangle$ the inner product on $C(M, \mathbb{R}^n)$ so that if p is a price and x is a consumption plan, $p, x \in C(M, \mathbb{R}^n)$, then the value of x at price level p is given by

$$\langle p, x\rangle = \int_M \langle p(t), x(t)\rangle \, dt,$$

with the standard inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$.

2.2.6 Individual demand functions

The individual demand functions $f_i : S \times (0, \infty) \to X$ of each agent i is the solution to the consumer's optimization problem so that

$$f_i(p, y) = arg\left[\max_{\langle p,x\rangle = y} U_i(x)\right].$$

It can be shown that for each agent i her individual demand function f_i is a differentiable map with differentiable inverse.

2.2.7 An exchange economy

In this paper we assume that preferences are fixed, so that the only parameters defining an economy are the initial endowments. Denote a generic economy by $\omega = (\omega_1, \ldots, \omega_I) \in \Omega = X^I$. For a fixed economy $\omega \in \Omega$ the **aggregate excess demand function** is a map $Z_\omega : S \to C(M, \mathbb{R}^n)$ defined by

$$Z_\omega(p) = \sum_{i=1}^{I} \left(f_i\left(p, \langle p, \omega_i\rangle\right) - \omega_i\right).$$

We also define $Z : \Omega \times S \to C(M, \mathbb{R}^n)$ by the evaluation

$$Z(\omega, p) = Z_\omega(p).$$

Because of Walras' law, it can be shown that the excess demand function satisfies $\langle p, Z_\omega(p)\rangle = 0$ for all $p \in S$.

2.2.8 The equilibrium set

We will close this section by defining the equilibrium set. This is done in the same way as in finite dimensions.

Definition 1. We say that $p \in S$ is an **equilibrium** of the economy $\omega \in \Omega$ if $Z_\omega(p) = 0$. We denote the **equilibrium set**

$$\Gamma = \{(\omega, P) \in \Omega \times S : Z(\omega, p) = 0\}.$$

The goal of this paper is to study some properties of the structure of the equilibrium set Γ.

3 Topological properties of the infinite equilibrium manifold

In our previous work [10] we began a systematic study of the infinite equilibrium set with separable utilities on a continuous consumption space. We established the following result.

Theorem 1. *[10] The equilibrium set* Γ *is a* C^∞ *Banach manifold and the natural projection map* $\pi : \Omega \times S|_\Gamma \to \Omega$ *is a* C^∞ *map.*

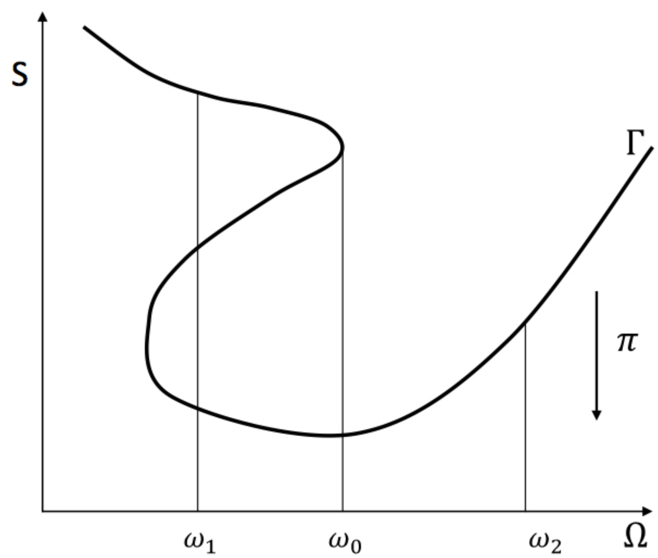

Figure 1: The equilibrium manifold Γ.

The goal of this section is to establish some global topological properties of the infinite equilibrium manifold. It turns out that its topology can be studied á la Balasko [4]. Our first result is presented in Theorem 2 which shows that the equilibrium manifold is made of linear fibers given by the equations that define it. This implies that the manifold will be arc-connected and simply connected. This is the infinite-dimensional analogue of Theorem

1 of [4], who shows that in finite dimensions Γ is also arc-connected and simply connected.

Recall (cf. [5]) that a topological space is arc-connected if it is always possible to link two arbitrarily chosen points of this space by a continuous path. Suppose that p is a price, ω the parameter that defines an economy, that (p,ω) is an element of the equilibrium manifold that describes a current equilibrium state and that (p',ω') an exogenously determined new equilibrium that is to be achieved at a future. Every path connecting these two equilibria is the mathematical expression of some economic policy. Connectedeness of the equilibrium set then proves that there is a continuous economic policy linking equilibria (p,ω) and (p',ω') while arc-conenctedeness expresses that any two such economic policies can be continuously deformed one into the other.

Theorem 2. *The infinite equilibrium manifold is contractible. In particular it is arc-connected and simply connected.*

Proof. Consider the map $f : S \times (\mathbb{R}_{++})^I \to S \times \Omega$ given by

$$f(p, w_1, \dots, w_I) = (p, f_1(p, w_1), \dots, f_I(p, w_I))$$

and the map $\phi : S \times \Omega \to S \times (\mathbb{R}_{++})^I$ given by

$$\phi(p, \omega_1, \dots, \omega_I) = (p, \langle p, \omega_1 \rangle, \dots, \langle p, \omega_I \rangle)\,.$$

Finally, let ψ be the restriction of ϕ to Γ. The proof then follows line by line the proof of Theorem 1 in [4]. □

4 A result on global uniqueness of equilibria

In this section we give a result on global uniqueness of equilibria. Some results on global uniqueness for infinite economies have been provided by Dana [13] for a different class of consumption spaces and by the author in [11] by constructing an index theorem. First recall the notion of regular and critical economies.

Definition 2. We say that an economy is **regular** (resp. **critical**) if and only if ω is a regular (resp. critical) value of the projection $pr : \Gamma \to \Omega$.

Theorem 3 below shows how global uniqueness of equilibria is intrinsically related to the existence of critical equilibria but also to the properness of the projection map. It is the infinite-dimensional analogue of Theorem 5.2 in [4].

Theorem 3. *For every smooth infinite economy ω to have a unique equilibrium it is necessary and sufficient that (i) there are no critical economies and (ii) a compact set of economies has a compact set of equilibrium prices.*

There is an equivalent formulation of Theorem 3 in terms of the projection map.

Theorem 4. *The projection map $\pi : \Gamma \to \Omega$ is a diffeomorphism if and only if π is proper and $D\pi : T\Gamma \to T\Omega$ is surjective.*

Proof of Theorem 3: From Theorem 2 above, the infinite equilibrium manifold Γ is connected. Also notice that Ω is simply connected as it simply is an open neighborhood of cross products of $C^{++}(M, \mathbb{R}^n)$.

First suppose that $\pi : \Gamma \to \Omega$ is a diffeomorphism. Then the Frèchet derivative of π is surjective everywhere; so every infinite economy is regular. But also, by assumption, the Walras correspondence π^{-1} is a continuous map so it must map a set of compact economies to a set of compact equilibrium prices. Hence π is proper.

Conversely, now suppose that (i) there are no critical economies and (ii) a compact set of economies has a compact set of equilibrium prices. We want to show that π is a diffeomorphism. Since there are no critical economies, the implicit function theorem between Banach spaces guarantees that the inverse is Frèchet differentiable. All we need to show then is that is a bijection.

Since π is proper, we can use a result of Palais [21] that a proper map sends closed sets into closed sets, i.e. $\pi(\Gamma)$ is closed. But also, since there are no critical economies, π is a local homeomorphism so it also sends open sets to open sets, i.e. $\pi(\Gamma)$ is open. Hence $\pi(\Gamma)$ is an open, closed and nonempty subset of Ω. So $\pi(\Gamma) = \Omega$. This shows that π is surjective.

We now show that π is injective. Consider two points γ_1, γ_2 in the equilibrium manifold Γ such that $\pi(\gamma_1) = \pi(\gamma_2) = \omega$. Since Γ is connected, we can consider a path $\alpha(t)$ in Γ connecting γ_1 to γ_2. Then $\pi \circ \alpha(t)$ is a loop in Ω based in ω. We also know that Ω is simply connected, so we may use a homotopy $F(s,t)$ such that $F(0,t) = \pi \circ \alpha(t)$ and $F(1,t) = \omega$. Since we have seen that π is is surjective, proper and a local homeomorphism from Γ to Ω, then by a result of Ho ([18], p.239), π must be a covering projection. And every covering projection has the homotopy lifting property property ([16], p.60). So there has to be a unique lifting $\tilde{F}(s,t)$ of $F(s,t)$ with $\tilde{F}(0,t) = \alpha(t)$. The lift of $\tilde{F}(1,t)$ must be a connected set containing both γ_1 and γ_2. But $\pi^{-1}(\omega)$ is discrete, so $\gamma_1 = \gamma_2$. $\square$

5 The projection map is Fredholm of index zero

We provide in this section a quick summary of the definitions of Fredholm index theory. We remind the reader that *Fredholmness* is a property on functions that allows us to extend to infinite dimensions some results of differential topology. Loosely speaking, a map is Fredholm if it derivative is almost invertible, i.e., if it is invertible up to compact perturbations. This notion was introduced by Smale in [23].

More precisely, a linear **Fredholm operator** is a continuous linear map $L : E_1 \to E_2$ from one Banach space to another with the following properties:

1. dim ker $L < \infty$;
2. range L is closed;
3. coker $L = E_2/\text{range}L$ has finite dimension.

If L is a Fredholm operator, then its **index** is defined to be equal to dim ker$L -$ dim coker$L$, so that the index of L is an integer.

A **Fredholm map** is a C' map $f : M \to V$ between differentiable manifolds locally like Banach spaces such that for each $x \in M$ the derivative $Df(x) : T_x M \to T_{f(x)} V$ is a Fredholm operator. The **index** of f is defined to be the index of $Df(x)$ for some x. If M is connected, this definition does not depend on x.

In our previous work [10] we have shown that the excess demand function $Z_\omega : S \to C(M, \mathbb{R}^n)$ of economy $\omega \in \Omega$ is a Fredholm map of index zero. Here we prove an equivalent result for the natural projection map. For completeness sake, we can mention that any smooth map between finite-dimensional spaces is a Fredholm map.

Theorem 5. *The map $\pi : \Gamma \to \Omega$ is Fredholm of index zero.*

Proof. The projection $\pi : \Gamma \to \Omega$ is Fredholm by a simple application of ([1], p.48). The index is constant across Γ since we have shown that it is connected. □

6 Critical economies and the number of equilibria

Let $B \subset \Gamma$ denote the set of critical prices and $\Sigma \subset \Omega$ be the set of critical economies. Now that we have remembered the notions of Fredholm index theory we can mention that Theorems 3 and 4 could be rephrased in a third equivalent way.

Theorem 6. *In order for every smooth infinite economy to have a unique equilibrium it is necessary and sufficient that (i) a compact set of economies has a compact set of equilibrium prices, (ii) the projection map* $\pi : \Gamma \to \Omega$ *is a Fredholm map of index zero and (iii)* $B = \emptyset$.

In finite dimensions, it is well known that, away from critical economies, prices vary continuously as functions of initial endowments (see figure below). This is the result obtained by Debreu in [14]. Theorem 7 below shows the extension of this result to infinite dimensions.

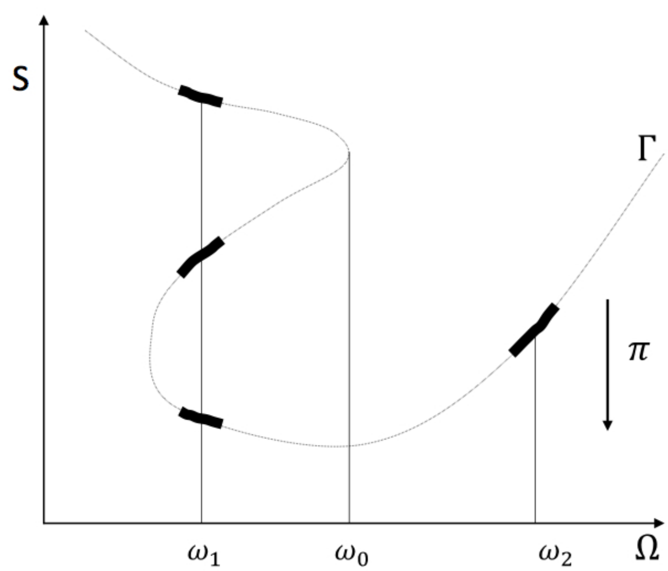

Figure 2: About ω_1,ω_2, prices vary continuously as functions of initial endowments. This is not the case for the critical economy ω_0.

Theorem 7. *Let Σ denote the singular set in Ω. Then $\pi : \Gamma - \pi^{-1}(\Sigma) \to \Omega - \Sigma$ is a covering space map.*

Proof. Since π is proper then by a result of Palais [21] it sends closed sets to closed sets. Hence the set of critical economies $\pi(B) = \Sigma$ is closed and therefore $\Gamma - \pi^{-1}(\Sigma)$ and $\Omega - \Sigma$ are both open.

The idea now is to show that every compact set of regular economies has a compact set of equilibrium prices. That is, to show that $\pi : \Gamma - \pi^{-1}(\Sigma) \to \Omega - \Sigma$ is a proper map. If this is the case, π will be both an open and a closed map and hence surjective. Hence, it will be a covering space map.

Let K be compact subset of $\Omega - \Sigma$. Then K is also compact in Ω. But $\pi^{-1}(K) \subset \Gamma - \pi^{-1}(\Sigma)$ and so it is compact in this set. $\square$

The previous result also shows that if the singular set Σ is small enough, the surjectivity of π is guaranteed and so we can focus on the study of π as a covering map and its behaviour on the critical set. It turns out that critical prices are removable in the following sense.

Theorem 8. *Let B be the set of critical equilibrium prices in Γ. Then, isolated critical prices are removable, i.e., if $p \in B$ is isolated in B then π is a local homeomorphism about p.*

Proof. The proof is basically Theorem 4 of [22]. $\square$

Our previous result shows that at isolated critical prices, locally we have $\pi^{-1}(\pi(B)) = B$. Our final result shows a globalisation of this result.

Theorem 9. *If the set of critical prices B is the countable union of compact sets, then $\pi^{-1}(\pi(B)) = B$ and π is a global diffeomorphism of $\Gamma - B$ onto $\Omega - \Sigma$.*

Proof. The proof is a simple application of Theorem 6 of [22]. $\square$

7 Conclusions

In this paper we set to study examples of economies with a continuous consumption space. We concentrated in understanding the structure of the equilibrium set and proved results in several directions. We already knew from previous work that the equilibrium set had the structure of a Banach manifold, but additionally we showed that it is contractible (and in particular, arc-connected and simply connected) and that outside the singular set the projection map is a covering space map. We also showed that critical prices are removable and that, if critical prices are the countable union of compact sets, then the projection map is a diffeomorphism outside the critical set.

Similarly, we gave different formulations of the problem of global existence of equilibria, showing the close relationship of multiplicity of equilibria and the existence of critical equilibria, as well as properness and the Fredholm index of the projection map.

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A Appendix: Mathematical definitions.

Definition 3. A topological space is said to be **contractible** if the identity map $i_X : X \to X$ is homotopic to a constant map.

Definition 4. A **Banach space** $(X, \|\cdot\|)$ is a normed vector space (over the real numbers throughout) that is complete with respect to the metric $d(x, y) = \|x - y\|$.

Definition 5. A **Hilbert space** H is a vector space with a positive-definite inner product $\langle \cdot, \cdot \rangle$ that defines a Banach space upon setting $\|x\|^2 = \langle x, x \rangle$ for $x \in H$.

Definition 6. A **bounded linear functional** $h(x)$ defined on a Banach space X is a linear mapping $X \to \mathbb{R}$ such that $|h(x)| \leq K\|x\|_X$ for some constant K independent of $x \in X$. The set of all bounded linear functionals on X, denoted X^*, is called the **conjugate space** of X. It is a Banach space with respect to the norm $\|h\| = \sup|h(x)|$ over the sphere $\|x\|_X = 1$. If $(X^*)^* = X$, then the space X is called **reflexive**.

Definition 7. One says that a set M of a Banach space X is **compact set** if M is closed (in the norm topology) and such that every sequence in M contains a strongly convergent subsequence.

Definition 8. A linear operator L with domain X and range contained in Y, (X,Y Banach spaces) is a **bounded linear operator** if there is a constant K independent of $x \in X$ such that $\|Lx\|_Y \leq K\|x\|_X$ for all $x \in X$. The set of such maps for fixed X,Y is again a Banach space, denoted $L(X,Y)$ with respect to the norm $\|L\| = \sup\|Lx\|_Y$ for $\|x\|_X = 1$.

Definition 9. A linear operator $C \in L(X,Y)$ is called a **compact operator** if for any bounded set $B \subset X$, $C(B)$ is conditionally compact in Y. Bounded linear mappings with finite-dimensional ranges are automatically compact; and conversely, if X and Y are Hilbert spaces, then a compact linear mapping C is the uniform limit of such mappings.

Relevant properties of linear compact operators. Let $C \in L(X,X)$ be compact, and set $L = I + C$. Then

1. L has closed range;

2. dim ker$L$ = dim coker$L < \infty$;

3. there is a finite integer β such that $X = \ker(L^\beta) \oplus \mathrm{range}(L^\beta)$ and L is a linear homeomorphism of range(L^β) onto itself.

Definition 10. Let $f \in C^1(U,Y)$, $U \subset X$, X,Y Banach spaces. Then, $x \in U$ is a **regular point** for f if $f'(x)$ is a surjective linear mapping in $L(X,Y)$. If $x \in U$ is not regular, x is called **singular point**. Similarly, **singular values** and **regular values** y of f are defined by considering the sets $f^{-1}(y)$. If $f^{-1}(y)$ has a singular point, y is called a singular value, otherwise y is a regular value.

Definition 11. An operator $f \in C^0(X,Y)$ is said to be a **proper operator** if the inverse image of any compact set C in Y, $f^{-1}(C)$ is compact in X.

The importance of this notion resides in the fact that the properness of an operator f restricts the size of the solution set $S_p = \{x : x \in X, f(x) = p\}$ for any fixed $p \in Y$.

Definition 12. A map f between topological space X, Y is said to be a **proper map** if the inverse image of each compact subset of Y is a compact subset of X.