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Equivalence between input-output and value-added economies

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Abstract: I show that, under standard assumptions, input-output (or network) economies are equivalent to value-added ones. Using a generalized version of the model in Acemoglu et al. (Econometrica, 2012), I show that the degree of influence of a given sector is equal to its value added share. This occurs because --by using the input-output network-- the output of a given sector indirectly contributes to the production of the final consumption of the rest of the sectors, which constitutes the source of its value. Thus, value-added economies deliver the same aggregate response to sectoral shocks than input-output ones. Despite this equivalence, the Leontief multiplier, which applies to sales and gross output, is intact.

Keywords: Input-output; Degree of influence; Value added; Network economies; Sectoral shocks

JEL Classification: E10; E16; C67

Resumen: Muestro que, bajo supuestos estándar, las economías de insumo-producto (o de redes) son equivalentes a las de valor agregado. Usando una versión generalizada del modelo de Acemoglu et al. (Econometrica, 2012) muestro que el grado de influencia de un sector es igual a su participación en el valor agregado de la economía. Esto ocurre así porque --usando la red de insumo-producto-- la producción de cada sector indirectamente contribuye a la producción de consumo final del resto de los sectores, lo que constituye la fuente de su valor. Por lo tanto, las economías de valor agregado generan la misma respuesta agregada ante choques sectoriales que las de insumo-producto. A pesar de esta equivalencia, el multiplicador de Leontief, que aplica a las ventas y a la producción bruta, permanece intacto.

Palabras Clave: Insumo-producto; Grado de influencia; Valor agregado; Economías de redes; Choques sectoriales

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1 Introduction

A recent literature, based on input-output economies, focuses on the interaction of sector-level shocks and aggregate outcomes (e.g., Caliendo, Parro, Rossi-Hansberg, and Sarte (2014), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2013), Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012), Jones (2011), Horvath (2000), Horvath (1998), and Long Jr and Plosser, 1987). The results in this literature stress the fact that the sectors differ in terms of their “degree of influence”, which is determined by the inter-connections among sectors through input-output relationships. The basic idea is that sectors that are suppliers of an important number of industries in the economy play a central role in the input-output network, and have a high degree of influence.

A standard result in this literature is that the effect on aggregate GDP of a change in productivity of sector i is obtained by multiplying the log change in productivity, da_i , times the degree of influence of the sector, v_i , that is $dy = v_i da_i$. In contrast, standard multi-sector value-added economies conclude that the change in aggregate GDP is obtained by multiplying the change in productivity of the sector times its value added share, α_i , that is $dy = \alpha_i da_i$.

In this paper, I show that, under standard conditions on the specification of the technology, the degree of influence of a given sector is simply equal to its value added share. That is, I show that $v_i = \alpha_i, \forall i$. I also show that the allocation of primary inputs across sectors is the same in both models. Moreover, under one additional general assumption, I show that equilibrium GDP levels in both economies are the same. In summary, I show an equivalence between input-output and value-added multi-sector economies.

The result helps to identify the key economic channels through which sectoral productivity changes affect aggregate variables and to clarify the role played by the Leontief multiplier in such outcomes. One key assumption is that productivity improvements work through affecting the efficiency of primary inputs and not (directly) the efficiency of intermediate ones. Thus, the equivalence result does not mean that we should forget about input-output economies, quite the contrary, it highlights the importance of understanding the true nature of technological and productivity improvements at the sector level, including the efficiency of intermediate inputs.

To understand the intuition behind the equivalence result it is necessary to reconsider two important features of input-output economies. The first feature is the difference between

value added and final consumption at the sector level. While the sum across sectors of each of these concepts is equal to GDP, these need not be equal at the individual sector level. But, if final consumption is the only source of value in the economy, how can the value added of an industry be different from the value of its own final consumption?

The key observation is that in input-output economies, gross output of a given industry can be used either as a final consumption good, or as an input in the production of other industries. Thus, the output of each industry, indirectly contributes to the production of many other consumption goods through the input-output network. The source of the value added by a given sector consists on its overall contribution to aggregate consumption –not only to its own consumption good, but possibly to other consumption goods produced by the rest of the sectors. For some industries, value added is higher than its own final consumption, which means that the output of this industry is being used to produce other consumption goods in a relatively intensive way. In contrast, for other industries, value added is lower than final consumption, which means that the source of such extra consumption is in the production that takes place in other industries. As an extreme example take the corporate sector. This sector is not consumed by households, therefore all its production is used as intermediate inputs in other industries. Does this mean that its value added is zero? No. Value added is still positive, because the corporate sector still contributes indirectly to the production of other final goods.

Crucially, the way the output of sector i contributes to the production of the consumption goods of other sectors is through the use of such sector's output as an intermediate input in the economy. As a result, there must be a connection between the value added share, and the importance of the sector in the input-output network –which is given by the degree of influence. If a sector plays a central role in the input-output structure, it means that it is the source of a large share of final consumption expenditures, and of a large share of GDP.

To further illustrate the argument, I present in the paper an example where sector 1 is the only supplier of intermediate inputs in the economy. Thus, sector 1 plays a very important role in the input-output network, and it has a large degree of influence. The number of sectors is very large, and each has an equal share of final consumption expenditures, so this share tends to zero. Does this mean that the value added of the sectors is zero? Not at all. In the paper, I show that the value added of sector 1 is larger than its own final consumption. In fact, it is equal to its own final consumption plus a fraction $0 < \sigma < 1$ of the final consumption of the rest of the sectors. In contrast, the value added by each of the other sectors is smaller than

its own consumption, and it is equal to the remaining fraction $(1 - \sigma)$ of their corresponding final consumption. Thus, the value added share of sector 1 is large and equal to its degree of influence. Likewise, the value added share of each of the other sectors is small, and equal to their corresponding degrees of influence.

The second feature of input-output economies needed to understand the equivalence result refers to the existence of a multiplier. In the model, there is a multiplier that applies both to sales and to gross output. The sales multiplier works exactly as in Leontief (1986): in order to increase the final consumption expenditures of sector i by \$1 dollar, it is necessary to increase the sales of sector i by more than \$1 dollar.¹ Put it slightly differently, start with an original change of \$1 dollar in the sales of sector i , thus, given the multiplier, you will end up increasing its sales by more than \$1. Of course, the difference with the Leontief analysis is that here the original increase of \$1 dollar in sales of sector i has to be given by a change in exogenous parameters. The same multiplier applies to gross output: start with any exogenous change that originally increases gross output of sector i by 1%, then the final change in sector i 's gross output will end up being larger than 1%. Note also that this means that a generalized original increase of 1% in gross output (sales) in each and every sector, ends up increasing aggregate gross output (aggregate sales) by more than 1%.

What about the multiplier for changes in productivity? Does an original 1% increase in the productivity of a sector generate an increase in GDP that is larger than its value added share? This is equivalent to ask whether the value of the degree of influence is bigger than the value added share. To answer this question, the specification of technology crucially matters. In particular, the answer depends on how the given change in productivity affects gross output. To illustrate this, I distinguish between a technological change that preserves a balance between primary and intermediate inputs, and one that does not. The first one I call it Z-type technology, and the second one A-type technology. Also note that the importance of primary inputs in the production function is given by the coefficient $(1 - \sigma_i)$, while the remaining fraction σ_i represents the importance of intermediate inputs.

When technology is of the A-type (unbalanced) a change in productivity affects only the efficiency of the subset of primary inputs, and it does not directly affect the efficiency with which intermediates are used in the production process. Therefore, in this case, a 1% increase

¹The reason for this is that in order for such an increase in consumption expenditures to occur, it is necessary to increase the purchases (and the sales) of other sectors which i uses as intermediates. Since the other sectors also use sector i as an intermediate input, the final change in sales of sector i has to be larger than \$1 dollar in order to supply such amount of consumption expenditures.

in the productivity of sector i generates an original increase in gross output that is less than 1%. In fact, output originally increases by $(1 - \sigma_i)(1\%)$ as a result of this exogenous change. This quantity is then multiplied through the input-output network. In this case, the degree of influence of a sector is exactly equal to its value added share in GDP.

In contrast, when I consider a Z-type technology (balanced) a change of 1% in the productivity of sector i translates into an original change in gross output of exactly 1%, which is then multiplied. As a result, the aggregate effect of a 1% percentage change in productivity is bigger in this case than in the A-type technology, and the degree of influence is given by the sales to GDP ratio (the “Dommar weights”), which is bigger than the value added share. The same results, nonetheless, can still be obtained in a value added economy if the technology parameters are raised to a power given by the inverse of the ratio of value added to gross output.

Related literature. This paper relates to the recent literature on input-output economies in the context of general equilibrium. Acemoglu et al. (2012) and Acemoglu et al. (2013) consider input-output economies to study certain properties of business cycles and large economic downturns. In particular, they consider a world where technical change is unbalanced between primary and intermediate inputs (A-type). These papers dig into the origins of fluctuations with special emphasis on the role of input-output inter-connections. My paper shows that an alternative (and simpler) model that would deliver the same results is a value-added economy with Hicks-neutral technical change.

Jones (2011) considers an input-output economy with a Z-type technology. Note that in the value-added economy, technological change will naturally be unbalanced between primary and intermediate inputs (as there are no intermediate inputs). Thus, in principle, the economy considered by Jones is not equivalent to a value-added economy. Nonetheless, in the paper I show that a mechanical way to make the value-added economy deliver the same results as in Jones (2011) is to consider productivity parameters raised to a power that corresponds to the inverse of the share of value added in gross output.

Jones (2011) also studies the effect of distortions to the price of gross output at the sector level. In a companion paper (Leal, 2015), I measure distortions similar to the ones in Jones (2011) for a typical developing country, and quantify the importance of eliminating them. One key feature of these distortions is that they are isomorphic to Z-type technological

change, and thus, a given percentage change in distortions gets fully multiplied.²

2 The value-added economy

This section presents an economy based on the choice of quantities that are expressed in terms of value added. This is the model that modern day macroeconomists are more familiar with. The main goals are to study the determinants of the efficient allocation, and to arrive to an expression that relates changes in productivity of individual sectors with changes in aggregate output.

There are N sectors, where N representative firms produce different goods. These firms are not inter-connected through input-output relationships, because they produce output in terms of value-added. There is a representative household that consumes a composite of the N goods, and is endowed with H units of labor. Preferences are given by $u(C)$, and leisure is not valued. Labor can be allocated to either sector without incurring any relocation costs.

In each sector i , there is a representative firm that has access to the following production function: $y_i = A_i h_i$, where h_i are hours in sector i , and A_i is labor productivity. Note, again, that no intermediate inputs are used in the production of y_i . There is also a producer of the composite that uses goods from the N sectors to produce, and has access to technology $\tilde{Y}(x_1, \dots, x_N) = \prod_{i=1}^N x_i^{\alpha_i}$, with $\sum_{i=1}^N \alpha_i = 1$. Where the tilde is used to distinguish this quantity from the corresponding one in the input-output economy.

SOCIAL PLANNER Consider the social planner's problem. Provided that, the utility function of the household is increasing in C , the social planner's problem consists on maximizing the production of the composite, subject to technological and resource constraints:

$$\begin{aligned} & \max_{\{x_i, h_i\}} \left\{ \prod_{i=1}^N x_i^{\alpha_i} \right\} \\ & \quad s.t. \\ & x_i = A_i h_i, \quad i \in \{1, \dots, N\} \\ & \quad \sum_i h_i = H \end{aligned}$$

²In this companion paper, I show that the effect of distortions on aggregate output depends on the input-output structure.

which can be simplified as:

$$\begin{aligned} \max_{\{h_i\}} A \prod_{i=1}^N h_i^{\alpha_i}; \quad A = \prod_{i=1}^N A_i^{\alpha_i} \\ \text{s.t.} \quad \sum_{i=1}^N h_i = H \end{aligned} \quad (1)$$

Thus, the problem of the social planner is to allocate labor across the sectors in order to maximize the production of the composite, subject to a feasibility constraint on the supply of labor. Note that there is a basic trade-off: the more labor is allocated to sector i , the less labor is allocated to sector $j \neq i$. As usual, the social planner will allocate labor in sector i up to the point in which the marginal productivity of an extra hour is equal to the marginal cost. And the first order conditions are:

$$\alpha_i \tilde{Y}(x_1, \dots, x_2) (A_i h_i)^{-1} A_i = \alpha_j \tilde{Y}(x_1, \dots, x_2) (A_j h_j)^{-1} A_j, \quad \forall i \neq j \quad (2)$$

$$\sum_{i=1}^N h_i = H \quad (3)$$

As equation 2 shows, the cost of allocating more hours to sector i is the lost production in sector $j \neq i$ associated with the reduction in hours that this sector experiences. Thus, the cost of allocating one extra hour in sector i is precisely the marginal productivity of sector $j \neq i$; and the solution of the social planner's problem is to allocate labor in such a way that marginal productivity across sectors is equalized. We can keep simplifying the first order conditions above to solve for the efficient allocation of labor:

$$\begin{aligned} \alpha_i \left(\frac{Y}{h_i} \right) &= \alpha_j \left(\frac{Y}{h_j} \right) \\ \Leftrightarrow \frac{h_i}{h_j} &= \alpha_i / \alpha_j, \end{aligned}$$

adding up across j :

$$\begin{aligned}
\Leftrightarrow \sum_{j=1}^N h_j \alpha_j &= \sum_{j=1}^N h_j \alpha_i \\
h_i \sum_{j=1}^N \alpha_j &= \alpha_i \sum_{j=1}^N h_j \\
h_i &= \alpha_i H
\end{aligned} \tag{4}$$

Note that the efficient allocation of labor is independent of the level of productivity A_i , $i \in \{0, 1\}$, a result that relies on the homogeneity of the composite, and sectorial production functions. Intuitively, an increase in A_i does not affect the allocation of labor because it shifts both sides of equation 2, proportionally. The reason for this is that x_i and x_j exhibit some degree of complementarity due to the form of $Y(x_1, \dots, x_2)$.

With the efficient allocation of labor across sectors, we can find expressions for efficient consumption and output in each sector: $x_i = y_i = A_i h_i = A_i \alpha_i H$, as well as an expression for aggregate output in terms of only parameters:

$$\tilde{Y} = (A_1 \alpha_1)^{\alpha_1} (A_2 \alpha_2)^{\alpha_2} \dots (A_N \alpha_N)^{\alpha_N} H, \tag{5}$$

intuitively, since all sectors exhibit constant returns to scale in labor, the “aggregate production function” also exhibits this feature. A useful expression arrives when we take logs and collect the “alphas” in a vector: $\tilde{y} = \ln \tilde{Y} = \alpha' a + \alpha' \ln \alpha + \ln H$. Note that the vector of “alphas” captures the effect of changes in sectoral productivity on aggregate output:

$$d\tilde{y} = \alpha_i da_i \tag{6}$$

A 1% increase in A_i increases aggregate output in $(\alpha_i)(1\%)$. In this sense, the vector of alphas is analogous to the “vector of influence” in Acemoglu et al. (2012) defined in the context of an input-output economy. Note also that if all A_s , $s = 1, \dots, N$ simultaneously increase by 1% then aggregate output increases by $\sum_{s=1}^N (\alpha_s)(1\%) = 1\%$, thus there is no TFP “multiplier” (in contrast to the input-output economy in Jones (2011)).

The vector of alphas is also the vector of “nominal” value-added shares. To see this more clearly, it is necessary to look at the decentralized equilibrium.

DECENTRALIZED EQUILIBRIUM First, consider the problem of the household.

Household The problem is as follows:

$$\max u(\tilde{C}), \text{ s.t. } \tilde{C} = \tilde{w}H \quad (7)$$

Thus, provided that u is increasing in \tilde{C} the solution consists on consuming all income.

Composite Normalizing the price of the composite to 1, the problem of the composite producer is as follows:

$$\max_{\{x_i\}} \tilde{Y}(x_1, \dots, x_2) - \sum_{i=1}^N \tilde{p}_i x_i$$

The first order conditions are given by:

$$\begin{aligned} \alpha_i \frac{\tilde{Y}(x_1, \dots, x_2)}{x_i} - \tilde{p}_i &= 0 \\ \sum_{i=1}^N \tilde{p}_i x_i &= \tilde{Y}(x_1, \dots, x_2) \end{aligned}$$

These conditions, lead to the well-known result with Cobb-Douglas functions, that consumption shares are constant and equal to the coefficients of the production function:

$$\alpha_i = \frac{\tilde{p}_i x_i}{\sum_{i=1}^N \tilde{p}_i x_i} = \frac{\tilde{p}_i x_i}{\tilde{Y}}, \quad (8)$$

Note also that, as mentioned above, this implies that α_i is equal to the sectoral value-added share of sector i . Thus, in this economy, as equation 6 states, the influence of productivity A_i on aggregate output is determined by the share of value added α_i .

3 The input-output economy

Next, I present an input-output economy and obtain equilibrium allocations as well as an expression for aggregate output in terms of only parameters. I am specially interested in comparing the way sectorial productivity shocks affect aggregate output in this economy vs. the economy in section 2.

The model is a version of the one in Long Jr and Plosser (1987), Jones (2011), and Acemoglu et al. (2012). Consider an economy with N sectors where the supply of labor (H) is exogenous. A representative firm in each sector uses labor and intermediate inputs to produce. These intermediate inputs are provided by the N different sectors, and a given sector also uses its own output as input. I assume that the production function of a representative firm in sector i is represented by the following Cobb-Douglas technology³:

$$Q_i = (A_i h_i)^{(1-\sigma_i)} \prod_{j=1}^N d_{ij}^{\sigma_{ij}}, \quad (9)$$

where d_{ij} represents the intermediate demand that industry i makes from industry j , A_i represents a Hicks-neutral exogenous productivity term, and h_i is labor used in sector i . Also, I define $\sigma_i = \sum_{j=1}^N \sigma_{ij}$. Thus, I assume Constant Returns to Scale (CRS), without loss of generality.

A crucial assumption is that c_j units of the output of a given sector j can be used to consume, while the rest is used as intermediate inputs. Thus, the resource constraint of sector j is given by:

$$Q_j = c_j + \sum_{i=1}^N d_{ij}, \forall i = 1, \dots, N. \quad (10)$$

Additionally, the part of each sector's output that is used for consumption is combined according to the following function:

$$Y(c_1, \dots, c_N) = \mathcal{L} c_1^{\beta_1} c_2^{\beta_2} \dots c_N^{\beta_N}, \quad (11)$$

³While this choice might seem a bit restrictive, it is nonetheless, a specification widely used –due to its tractability– in the literature using input-output economies. Prominent papers using this technology are: Horvath (2000), Jones (2011), Acemoglu et al. (2012), and Caliendo, Parro, Rossi-Hansberg, and Sarte (2014).

where $\mathcal{L} > 0$ is a parameter that will be useful below when I find an expression for equilibrium output. Note that I have intentionally made a distinction between consumption in the economy in terms of value-added (x_i), and consumption of gross output here (c_i). Similarly, I have made a distinction between the coefficients in this production function (β_i) and the ones in the economy using value-added units (α_i). I also made a distinction between aggregate output in this economy Y , and aggregate output in the economy using value-added, \tilde{Y} .

SOCIAL PLANNER The problem of the social planner in this economy is to maximize the production of the composite $Y(c_1, \dots, c_N)$ subject to the resource constraints:

$$\begin{aligned} & \max_{\{\{d_{ij}\}, \{h_i\}\}} \mathcal{L} \prod_{i=1}^N c_i^{\beta_i} \\ & \text{s.t.} \\ & c_i = (A_i h_i)^{(1-\sigma_i)} \prod_{j=1}^N d_{ij}^{\sigma_{ij}} - \sum_{i=1}^N d_{ij}, \quad \forall i = 1, \dots, N. \\ & \sum_{i=1}^N h_i = H. \end{aligned}$$

Note that the choice variables consists on the allocation of labor and intermediate inputs across sector. It is useful to break-down this problem into two sub-problems: first, the problem of choosing the optimal allocation of intermediate inputs –given an arbitrary allocation of labor– and second, the problem of choosing the optimal allocation of labor across sectors –given the optimal allocation of intermediate inputs.

$$\begin{aligned} & \max_{\{h_i\}} \left\{ \max_{\{d_{ij}\}} \mathcal{L} \prod_{i=1}^N \left[(A_i h_i)^{(1-\sigma_i)} \prod_{j=1}^N d_{ij}^{\sigma_{ij}} - \sum_{i=1}^N d_{ij} \right]^{\beta_i} \right\} \\ & \text{s.t.} \\ & \sum_{i=1}^N h_i = H \end{aligned}$$

By substituting the first order conditions of the first stage of the problem into its objective, it can be shown that this simplifies to:⁴

$$\begin{aligned} \max_{\{h_i\}} & \left\{ \bar{\mathcal{A}} A \prod_{i=1}^N h_i^{v_i} \right\} & (12) \\ & s.t. \\ & \sum_{i=1}^N h_i = H \end{aligned}$$

where v_i is known as the degree of influence of sector i and is a function of primitive parameters (see below). Similarly, $\bar{\mathcal{A}}$ is a function of parameters and is defined in the appendix. Note the striking similarities between this problem and problem 1 above. This implies that the the social planners' problem of the I-O economy admits a representation similar to the problem of the VA economy. To make both problems identical, it remains to show that: 1) the degree of influence of a sector in the IO economy (v_i) can also be interpreted as its value added share; and 2) $\bar{\mathcal{A}} = 1$. For such a purpose, it is more useful to look at the equilibrium conditions of the decentralized equilibrium. Furthermore, the equilibrium conditions below will allow us to clarify the role played by the Leontief multiplier in the IO economy.

DECENTRALIZED EQUILIBRIUM I start by describing the problems of the agents interacting in equilibrium: the representative household, the producer of each sector, and the producer of the composite good. Later, I analyze the equilibrium conditions and compare the two economies. I also analyze the role played by the Leontief multiplier.

Problem of the representative household. The household consumes units of the composite and the problem faced is similar to the one in 7:

$$\max_{\{C\}} \{u(C)\}$$

⁴I am grateful to an anonymous referee for pointing this out and for providing the proof. Th proof is available upon request.

$$s.t. C = wH \quad (13)$$

where C is aggregate consumption and w is the price of labor. Provided u is increasing, the solution for this problem is trivial: the household will consume all the available income. Note that I have intentionally used C to describe aggregate consumption in this economy, while I have used \tilde{C} to describe aggregate consumption in the economy using value-added.

Problem of the composite producer. The problem of the composite producer consists on choosing $\{c_i\}$, taking $\{p_i\}$ as given, to solve:

$$\max_{\{c_i\}} \left\{ \mathcal{L} c_1^{\beta_1} c_2^{\beta_2} \cdots c_N^{\beta_N} - \sum_{i=1}^N p_i c_i \right\}.$$

The first order conditions are given by:

$$\beta_i(Y/c_i) - p_i = 0 \Leftrightarrow \beta_i = \frac{p_i c_i}{Y}, \forall i. \quad (14)$$

There are two important results from the optimal conditions of this problem. The first is that the consumption shares in GDP are constant and equal to the coefficients of the production function, the second one is that aggregate output is the sum of consumption expenditures: $Y = \sum_{i=1}^N p_i c_i$. Note that condition 14 is similar to the one in 8, but they refer to very different concepts. While equation 8 refers to shares of value-added in GDP, equation 14 refers to shares of final consumption.

Value added vs. final consumption expenditures. To see the important difference between sectoral value added and sectoral final consumption, define nominal value added in sector i as:

$$NVA_i = p_i Q_i - \sum_{j=1}^N p_j d_{ij}, \quad (15)$$

and compare this with the resource constraint multiplied by p_j :

$$p_j c_j = p_j Q_j - \sum_{i=1}^N p_j d_{ij}, \forall i = 1, \dots, N. \quad (16)$$

Notice that, it is not necessarily true that $p_i c_i$ is equal to the value-added of sector i . In fact, $p_i c_i$ and NVA_i will, in general, differ. An inspection of equations 15 and 16, leads to conclude that: $NVA_s = p_s c_s \Leftrightarrow \sum_{j=1}^N p_j d_{sj} = \sum_{i=1}^N p_s d_{is}$. The last equality does not usually hold in the data: the value spent by a given sector s on intermediate inputs is not equal to the value of intermediate inputs supplied by s .

While individual (sectoral) consumption expenditure differs from individual nominal value added, the sum of consumption expenditures across sectors and the sum of sectoral value added across sectors, both, add-up to GDP. To see this, consider the same two equations (15 and 16), but adding-up across the relevant indices. For 15 we have:

$$\sum_{i=1}^N NVA_i = \sum_{i=1}^N p_i Q_i - \sum_{i=1}^N \sum_{j=1}^N p_j d_{ij}, \quad (17)$$

while for equation 16, we have:

$$\sum_{j=1}^N p_j c_j = \sum_{j=1}^N p_j Q_j - \sum_{j=1}^N \sum_{i=1}^N p_j d_{ij}, \forall i = 1, \dots, N, \quad (18)$$

combining the two, we have:

$$\sum_{i=1}^N NVA_i = \sum_{j=1}^N p_j c_j = Y,$$

where the last equality follows from condition 14 in the problem of the composite.

Problem of the representative firm in sector i . The problem of the representative firm in sector i is given by:

$$\max_{h_i, \{x_{ij}\}} \left\{ p_i (A_i h_i)^{(1-\sigma_i)} \prod_{j=1}^N d_{ij}^{\sigma_{ij}} - w h_i - \sum_{j=1}^N p_j d_{ij} \right\}$$

and the first order conditions (FOCs) are as follow:

$$(1 - \sigma_i) \frac{p_i Q_i}{h_i} = w, \forall i \quad (19)$$

$$\sigma_{ij} \frac{p_i Q_i}{d_{ij}} = p_j, \forall i, j \quad (20)$$

An important feature of the firms' problem is that value-added is proportional to the value of gross output:

$$NVA_i = (1 - \sigma_i) p_i Q_i \quad (21)$$

To see why, note that by 20, we have: $\sum_{j=1}^N p_j d_{ij} = \sum_{j=1}^N \sigma_{ij} p_i Q_i = p_i Q_i \sum_{j=1}^N \sigma_{ij} = \sigma_i p_i Q_i$.

Equilibrium. With this, we can provide a definition of competitive equilibrium. A competitive equilibrium consists on quantities $\{h_i, x_{ij}, c_i\}$; and prices $\{p_j\}$ and $w, \forall i, j = 1, \dots, N$; such that:

1. $\{c_i\}$ solves the composite producer problem at the equilibrium prices.
2. h_i , and $\{d_{ij}\}$ solve sector's i producer problem at the equilibrium prices.
3. Markets for labor, and goods $j = 1, \dots, N$ clear.

3.1 Equilibrium Characterization

Leontief and multipliers. One first important feature of the equilibrium is the existence of a multiplier of sales very much in the spirit of Leontief (1986). Leontief argued that in order to increase final consumption expenditures by \$1 dollar in sector i , it is necessary to increase sales in sector i by more than \$1 dollar. The reason for this is that in order for such an increase in consumption expenditures to occur, it is necessary to increase the purchases (and the sales) of other sectors which i uses as intermediates. Since the other sectors also use sector i as an intermediate input, the final change in sales of sector i has to be larger than \$1 dollar in order to supply such amount of consumption expenditures.

In the model, this multiplier also exists because the purchases of intermediate inputs are proportional to sales. Define sales by $s_j = p_j Q_j$, and final consumption expenditures by $f_j = p_j c_j$, it can be shown that, in the model, the following equilibrium equation holds:

$$s_j = f_j + \sum_{i=1}^N \sigma_{ij} s_i \quad (22)$$

In vector form:

$$\begin{aligned} \mathbf{s} &= \mathbf{f}' + \mathbf{sB} \\ \mathbf{s} &= \mathbf{f}'(\mathbf{I} - \mathbf{B})^{-1} \end{aligned}$$

where \mathbf{B} is the $N \times N$ matrix of technical coefficients with typical element σ_{ij} , and $(\mathbf{I} - \mathbf{B})^{-1}$ is known as the Leontief inverse, which maps changes in final consumption expenditures on changes in sector sales. A typical element l_{ij} of the Leontief matrix gives the change in sales of sector j needed to achieve an increase in final consumption expenditures in sector i of \$1 dollar. This effect captures all first and second round downstream effects that occur through the input-output network.

Since consumption expenditures are just a fraction of sales of a given sector, this argument can be generalized for any original exogenous change of \$1 dollar in the sales of sector i . Furthermore, as it is shown in the appendix, the argument can also be applied to (log) gross output, not only to sales: start with an originally exogenous change in (log) gross output of sector i of magnitude dq_i , thus, due to the multiplier, the final change in sector i 's gross output will end up being larger than the original amount dq_i . As a result, any change in exogenous parameters that directly affects gross output in a given sector, will be multiplied. In equilibrium, gross output is given by:

$$\begin{aligned} \mathbf{q} &= \mathcal{Q} + \mathbf{Bq} \\ \mathbf{q} &= (\mathbf{I} - \mathbf{B})^{-1} \mathcal{Q} \end{aligned}$$

Any change in the exogenous parameters included in \mathcal{Q} , in the right hand side of the above equation (see the appendix), will be multiplied through the input-output network. However, it is important to keep track of what gets multiplied. In order for the multiplier to fully work, we must apply the same logic as in the case of consumption expenditures and sales. That is, we need an exogenous change in parameters that creates an original change in (log) gross output of 1 unit, just like a \$1 dollar increase in consumption expenditures creates an original increase of \$1 dollar in sales, which then gets multiplied.

Given that consumption is a constant fraction of gross output in each sector (see the appendix), percentage changes in gross output will be equivalent to percentage changes in final consumption. Thus, the Leontief inverse also gives the percentage changes in final consumption expenditures associated to the changes in exogenous parameters. As a result we can obtain the total effect on log GDP of these exogenous changes by adding up the corresponding consumption changes using the appropriate weights. In this case, these weights are given by the final consumption expenditure shares β_i ⁵. Thus, the percentage change in GDP of an exogenous original change in the gross output of sector i of 1% will be given by: $m_i = \sum_{j=1}^N \beta_j l_{ij}$. Or in vector notation:

$$\mathbf{m} = \boldsymbol{\beta}'(\mathbf{I} - \mathbf{B})^{-1}.$$

where $\boldsymbol{\beta}$ is a $N \times 1$ vector of final consumption expenditures shares, and \mathbf{m} is the $1 \times N$ vector of multipliers. It turns out that these multipliers are closely related to sales, and are fully captured by a simple object known as the ‘‘Dommar weights.’’⁶ These weights are equal to sector’s sales divided by GDP. To see why, take equation 22, and divide it by Y :

$$\frac{s_j}{Y} = \frac{f_j}{Y} + \sum_{i=1}^N \frac{\sigma_{ij} s_j}{Y},$$

Now define $\tilde{s}_j = \frac{p_j Q_j}{Y}$, thus:

$$\tilde{s}_j = \beta_j + \sum_{i=1}^N \sigma_{ij} \tilde{s}_i$$

⁵To see why, note that equation 11 in logs is equal to: $y = \sum_{i=1}^N \beta_i \ln c_i$

⁶Domar (1961) argued that these are the appropriate weights to use when obtaining a measure of aggregate productivity from sector level productivity data. In the present model, these weights arrive naturally by combining the resource constraint and first order conditions.

which in vector notation is equal to the vector of multipliers $\tilde{\mathbf{s}} = \mathbf{m} = \beta'(\mathbf{I} - \mathbf{B})^{-1}$. One final and important property of the Dommar weights is that these sum to more than 1.

Aggregate output. By substituting first order conditions in the definition of aggregate output I arrive to the following expression of equilibrium aggregate output (see the appendix):

$$Y = \mathcal{A}H = \tilde{\mathcal{A}}(A_1v_1)^{v_1}(A_2v_2)^{v_2}\dots(A_Nv_N)^{v_N}H, \quad (23)$$

where $\tilde{\mathcal{A}}$ is a scalar that depends on parameters (see the Appendix) and v_i is a typical element of the $1 \times N$ “vector of influence” \mathbf{v} (see Acemoglu et al. (2012)). A typical element of \mathbf{v} is called the “degree of influence” of sector i because it gives the effect of a 1% increase in the productivity of sector i (A_i) on aggregate output. To see this write 23 in logs:

$$y = \ln Y = \ln \tilde{\mathcal{A}} + \mathbf{v}\mathbf{a} + \mathbf{v}\ln\mathbf{v}' + \ln H,$$

where \mathbf{a} is a $N \times 1$ vector that collects the log productivity parameters, with typical element $a_i = \ln A_i$. Thus, the derivative of log output with respect to a_i is:

$$dy = v_i da_i.$$

One interesting feature of the vector of influence emphasized in previous studies is that it depends on the input-output relationships across sectors. In fact, the degree of influence of sector i is equal to the share of value added in gross output times the multiplier:

$$v_i = (1 - \sigma_i)m_i.$$

Thus, a 1% increase in productivity of sector i first gets adjusted by the factor $(1 - \sigma_i)$, and then it gets multiplied. This reflects the fact that in the current specification of the input-output economy, productivity only affects the production efficiency of primary inputs, and it does not affect the efficiency of intermediate inputs.

Comparing the two economies. It turns out that the vector of influence is equal to the vector of industrial value added shares:

$$v_i = (1 - \sigma_i)m_i = \frac{(1 - \sigma_i)p_i Q_i}{Y} = \frac{NVA_i}{Y}.$$

The reason for this is that, as equation 21 shows, value added is proportional to sales in each sector. Note that the same value added shares are given by α_i in the economy of section 2, or, in other words, that: $\alpha_i = v_i, \forall i$. Note, finally, the similarity between equation 5 and equation 23, which give aggregate output in both economies. Given that I have shown that $\alpha_i = v_i, \forall i$, at this point, the only difference between the two equations relies on the parameter $\bar{\mathcal{A}}$. However, by appropriately choosing the value of \mathcal{L} it is possible to set $\bar{\mathcal{A}} = 1$, which will make both economies achieve the same level of aggregate output (see the Appendix).

In addition, the two economies achieve also the same labor allocations. To see this, note that I can write the labor allocation in the input-output economy in the following way:

$$h_i = v_i H, \forall i$$

which is the same as equation 4 from section 2.

What is the logic behind this equivalence? The logic behind this result follows from the analysis in section 3. The value added generated in a single sector, is used for the final consumption expenditures of many sectors, and the way this value reaches other sectors is through the use of such sector output as intermediate input in the production of other sectors. Thus, there is an equivalence between the value added share of the sector, and the importance of the sector in the input-output network. As a result, any exogenous change that increases value added in a sector, will have a 1 to 1 impact on final consumption expenditures.

Balanced vs. unbalanced technical change. One interesting case is when instead of having a technology parameter that only affects the efficiency of primary inputs, we have a technology parameter that affects the efficiency of all inputs. In this case, I re-define the gross output function as:

$$Q_i = Z_i (h_i)^{(1-\sigma_i)} \prod_{j=1}^N d_{ij}^{\sigma_{ij}}, \quad (24)$$

Note that a change in Z_i affects simultaneously and in a balanced way the efficiency of all inputs.⁷ I distinguish between this type of technological change which is balanced between intermediate and primary inputs, and the unbalanced technological change in equation 9. I call A-type technological change to the latter, and Z-type to the former. One important point is that the economy in section 2, by definition, can not affect the productivity of intermediate inputs, thus, the comparable case in the input-output economy is the A-type technology.

Note also that there is a mapping between the two types of technology parameters. In particular, note that $A_i = Z_i^{\frac{1}{1-\sigma_i}}$. Moreover, since $0 < \sigma_i < 1$, the exponent $\frac{1}{1-\sigma_i} > 1$. Thus, a 1% increase in Z-type productivity is equivalent to a more than 1% increase in A-type productivity. As a result, the aggregate effects of comparable percentage changes in Z-productivity are bigger than in A-productivity.

When technology is Z-type, equilibrium aggregate output can be written as:

$$Y = \mathcal{A} \left(\prod_{i=1}^N Z_i^{m_i} \right) \left(\prod_{i=1}^N v_i^{v_i} \right) H$$

and the vector of influence in this case is given by:

$$dy = m_i dz_i$$

where $z_i = \ln Z_i$. Thus, in the case of Z-type productivity, an original change of 1% in productivity gets fully multiplied through the input-output network. What turns out to be crucial is to keep track of what exactly gets multiplied when a productivity change occurs. It is at this point where the distinction between A-type and Z-type of technological change matter. When Z-type technology is in place, a 1% increase in productivity of sector i leads to an original increase of 1% in gross output, which is multiplied accordingly. Note that in this case, a change in productivity alters the efficiency with which all inputs are used in the production function, including intermediate ones. However, when the technology is of the A-type the original response of gross output is less than 1%, because the productivity parameter in this case is raised to the power $(1 - \sigma_i)$. In fact, the original response of log gross output is adjusted by this power. As a result, the multiplier applies only to an original change in gross

⁷I could also have written the above equation as $Q_i = (Z_i h_i)^{(1-\sigma_i)} \prod_{j=1}^N (Z_j d_{ij})^{\sigma_{ij}}$, which better reflects the key property of this technology.

output of $(1 - \sigma_i)(1\%)$ and the final change in gross output as a response to a 1% change in productivity is lower. A change in productivity in this case, does not affect the efficiency with which electricity or other intermediates are used in the production function, it affects only the efficiency of a subset of inputs: the primary ones.

4 Simple example

In this section, I outline a simple example to gain intuition on the main argument of the paper. The basic set-up of this example is taken from Acemoglu et al. (2013). First, I re-write the vector of influence as:

$$v_j = (1 - \sigma_j)\beta_j + (1 - \sigma_j) \sum_{s=1}^N \left(\frac{v_s}{1 - \sigma_s} \right) \sigma_{sj}. \quad (25)$$

This way of writing the elements of the vector of influence is useful to compute the value of individual elements. It also shows that the degree of influence of a given sector j depends on the degree of influence of the sectors to which j connects downstream more strongly. Note that the second term is a weighted average of the degree of influence of the rest of the sectors. The weights of this average correspond precisely to the importance of sector j in the production of the rest of the sectors ($\sigma_{sj}/(1 - \sigma_s)$).

I next assume that the sectors share common parameters: a) all sectors are equally important in the composite production, i.e. $\beta_i = 1/N, \forall i$; and, b) all sectors have the same share of value added in gross output, i.e. $\sigma_i = \sigma, \forall i$. Furthermore, I assume that sector 1 is the only input supplier in the economy. The rest of the sectors are producers of final goods and do not supply any intermediate inputs. This implies that $\sigma_{ij} = 0, \forall j \neq 1$, and that $\sigma_{i1} = \sigma, \forall i$.

How does the degree of influence look like for the sectors in this economy? Using equation 25 above, I get $v_j = (1 - \sigma)(1/N)$, for sectors $j \neq 1$; while for sector 1, I get $v_1 = (1 - \sigma)/N + \sigma$. One illustrating exercise is to compute the degree of influence in each sector when $N \rightarrow \infty$. In this case, the final consumption in each sector is a very small fraction of total GDP (i.e. $p_i c_i / Y = (1/N) \rightarrow 0$). Taking limits, we get that $v_{j \neq 1} \rightarrow 0$, and $v_1 \rightarrow \sigma$. Thus, the influence of sector 1 is quite large compared to the individual influence of each of the rest of the sectors. The reason for this is that sector 1 is an important supplier in the economy, while the rest are not.

How does the influence of each sector compare with the value added shares? To illustrate this, let's compute the value added shares of each sector. For this, I use the fact that value added is a constant share of gross output combined with the resource constraint in each sector. For the case of sectors $j \neq 1$, these are only used as final consumption, thus, I get:

$$NVA_j = (1 - \sigma)p_j Q_j = (1 - \sigma) \left[p_j c_j + \sum_{i=1}^N p_j d_{ij} \right] = (1 - \sigma)p_j c_j = (1 - \sigma) \frac{1}{N} Y, \quad \forall j \neq 1, \quad (26)$$

where the fourth equality follows from substituting the first order conditions of the composite producer. Note that the share of value added of each of these sectors ($NVA_j/Y = (1 - \sigma)/N$) is exactly the same than the degree of influence obtained using equation 25. Following the same procedure, I obtain the following expression for sector 1:

$$\begin{aligned} NVA_1 &= (1 - \sigma)p_1 Q_1 = (1 - \sigma) \left[p_1 c_1 + \sum_{i=1}^N p_1 d_{i1} \right] = (1 - \sigma)p_1 c_1 + (1 - \sigma) \sum_{i=1}^N \sigma p_i Q_i \\ &= (1 - \sigma) \frac{1}{N} Y + \sigma \sum_{i=1}^N (1 - \sigma) p_i Q_i = \left((1 - \sigma) \frac{1}{N} + \sigma \right) Y, \end{aligned} \quad (27)$$

Note that equation 27 implies that the share of value added of sector 1 ($NVA_1/Y = (1 - \sigma)/N + \sigma$) is equal to its degree of influence obtained above. Next, consider what happens when $N \rightarrow \infty$. In this case, final consumption of the sectors is infinitely small, but since sector 1 is an important supplier in the economy, the value added share of sector 1 (as well as its degree of influence) is a positive number:

$$\frac{NVA_1}{Y} \rightarrow \sigma$$

In contrast, sectors $j \neq 1$ produce only final goods, thus their share of value added is infinitely small as consumption: $\frac{NVA_j}{Y} = (1 - \sigma)/N \rightarrow 0, \forall j \neq 1$. However, this does not mean, that the value added share of the sum of all sectors different than 1 is zero. Adding-up across all $j \neq 1$, and taking the limit, we have that:

$$\sum_{j \neq 1} \frac{NVA_j}{Y} = (1 - \sigma) \frac{N-1}{N} \rightarrow (1 - \sigma).$$

Note that the sum of the value added shares across all sectors is equal to one⁸. More interestingly, note how value added compares with consumption expenditures in each sector. For the case of a sector $j \neq 1$, we have that $NVA_j = (1 - \sigma)p_j c_j < p_j c_j$. This means that only a fraction $(1 - \sigma)$ of the final consumption in these sectors is supported by their own value added. In contrast, the value added of sector 1 is larger than its final consumption expenditures: $NVA_1 > p_1 c_1$. In fact, it is possible to show that the value added of sector 1 can be re-written in terms of only consumption as:

$$NVA_1 = p_1 c_1 + \sum_{j \neq 1}^N \sigma p_j c_j$$

Thus, the value added of sector 1 is used to support not only consumption of sector 1, but also a fraction σ of all other sector's consumption expenditures. In summary, the influence of sector 1 is large, but this influence is fully captured by its value added share.

5 Conclusion

In this paper, I show that, under plausible conditions on the specification of the technology, the degree of influence of a given sector is simply equal to its value added share. Which implies that all the input-output structure is captured by such shares.

A key feature of input-output economies is that the value added by a single sector, can be used to support the final consumption expenditures of many different sectors. Conversely, there may be sectors where the value added is not enough to purchase its own final consumption expenditures. The way the value added in a given sector i reaches other sectors is through the use of sector i 's output as an intermediate input. Thus, when a sector plays a central role in the input-output network, it also has a large value added share.

Despite the equivalence between input-output and value added economies, the multiplier of gross output is still present in input-output economies. This result emphasizes the need to keep track of what exactly is getting multiplied as a response to changes in exogenous parameters.

⁸ $\frac{NVA_1}{Y} + \sum_{j \neq 1} \frac{NVA_j}{Y} = \sigma + (1 - \sigma) = 1$

References

- ACEMOGLU, D., V. M. CARVALHO, A. OZDAGLAR, AND A. TAHBAZ-SALEHI (2012): “The network origins of aggregate fluctuations,” *Econometrica*, 80, 1977–2016.
- ACEMOGLU, D., A. OZDAGLAR, AND A. TAHBAZ-SALEHI (2013): “The network origins of large economic downturns,” Tech. rep., National Bureau of Economic Research.
- CALIENDO, L., F. PARRO, E. ROSSI-HANSBERG, AND P.-D. SARTE (2014): “The impact of regional and sectoral productivity changes on the US economy,” Tech. rep., National Bureau of Economic Research.
- DOMAR, E. D. (1961): “On the measurement of technological change,” *The Economic Journal*, 709–729.
- HORVATH, M. (1998): “Cyclical and sectoral linkages: Aggregate fluctuations from independent sectoral shocks,” *Review of Economic Dynamics*, 1, 781–808.
- (2000): “Sectoral shocks and aggregate fluctuations,” *Journal of Monetary Economics*, 45, 69–106.
- JONES, C. I. (2011): “Misallocation, economic growth, and input-output economics,” Tech. rep., National Bureau of Economic Research.
- LEAL, J. (2015): “Key sectors for economic development: A perspective from input-output linkages and cross-sector misallocation,” *Manuscript*.
- LEONTIEF, W. (1986): *Input-output economics*, Oxford University Press on Demand.
- LONG JR, J. B. AND C. I. PLOSSER (1987): “Sectoral vs. aggregate shocks in the business cycle,” *American Economic Review*, 77, 333–36.

Appendix

Derivation of equilibrium

The goal is to obtain expressions for labor, intermediate inputs, and output in terms of only parameters. First, I write the Dommar weights $m_i = p_i Q_i / Y$ in terms of only parameters,

because it will be useful for later.

Note that we can use the resource constraint to obtain an expression of the Dommar weights in terms of only parameters:

$$\begin{aligned}
 Q_j &= c_j + \sum_{i=1}^N d_{ij}, \forall j \\
 \text{multiply by } p_j &: p_j Q_j = p_j c_j + \sum_{i=1}^N p_j d_{ij}, \forall j \\
 \text{divide by } Y &: \frac{p_j Q_j}{Y} = \frac{p_j c_j}{Y} + \sum_{i=1}^N \frac{p_j d_{ij}}{Y}, \forall j \\
 \text{Use FOC} &: m_j = \beta_j + \sum_{i=1}^N \sigma_{ij} m_i \forall j
 \end{aligned}$$

In vector notation:

$$m = \beta' + mB,$$

where m is a $1 \times N$ vector of Dommar weights, β is a $N \times 1$ vector of consumption expenditures, and B is the matrix of technical coefficients with typical element σ_{ij} . Solving for m , I get:

$$m = \beta'(I - B)^{-1}.$$

Note that if the Dommar weights are multiplied by $(1 - \sigma_j)$ we get the “vector of influence” v_j , which is defined as:

$$v_j = (1 - \sigma_j)m_j,$$

note also that these correspond to the shares of value added, and by definition these add-up to 1: $\sum_i v_i = 1$.

With this at hand, we can find an expression for labor in terms of only parameters. Take the first order condition for labor:

$$\begin{aligned}
h_j &= (1/w)(1 - \sigma_j)p_jQ_j \\
\frac{h_j}{Y} &= (1/w)(1 - \sigma_j)\left(\frac{p_jQ_j}{Y}\right) \\
h_j &= (Y/w)(1 - \sigma_j)m_j \\
\frac{h_j}{H} &= \frac{(Y/w)(1 - \sigma_j)m_j}{(Y/w)\sum_{i=1}^N(1 - \sigma_i)m_i} \\
h_j &= (1 - \sigma_j)m_jH
\end{aligned}$$

where I have used in the last equation the fact that $\sum_{i=1}^N(1 - \sigma_i)m_i = 1$. Note that we can write equilibrium labor in terms of the vector of influence as:

$$h_j = v_jH.$$

Following a similar procedure, we can also find an expression for d_{ij} in terms of only parameters:

$$d_{ij} = \sigma_{ij}\left(\frac{m_i}{m_j}\right)Q_j$$

Thus, we can substitute the expressions we found for labor and for intermediate inputs into the definition of gross output:

$$Q_i = (A_i v_i H)^{(1 - \sigma_i)} \prod_{j=1}^N \left(\sigma_{ij} \frac{m_i}{m_j} Q_j\right)^{\sigma_{ij}}$$

taking logs:

$$\ln Q_i = (1 - \sigma_i)(\ln A_i + \ln v_i + \ln H) + \sum_{j=1}^N \sigma_{ij} \ln\left(\sigma_{ij} \frac{m_i}{m_j}\right) + \sum_{j=1}^N \sigma_{ij} \ln Q_j$$

in vector notation:

$$\mathbf{q} = \left\| \left\| (1 - \sigma_i) a_i \right\| + \left\| (1 - \sigma_i) \ln v_i \right\| + [1 - \sigma] \ln H + \left\| \sum_{j=1}^N \sigma_{ij} \ln \left(\sigma_{ij} \frac{m_i}{m_j} \right) \right\| \right\| + \mathbf{B} \mathbf{q}$$

where the notation $\|x_i\|$ is used to represent a $N \times 1$ vector with typical element x_i , $[1 - \sigma]$ is a $N \times 1$ vector with elements $(1 - \sigma_i)$, and q is a $N \times 1$ vector with elements given by $\ln Q_i$. Solving for q , we have:

$$q = (\mathbf{I} - \mathbf{B})^{-1} \left(\left\| \left\| (1 - \sigma_i) a_i \right\| + \left\| (1 - \sigma_i) \ln v_i \right\| + [1 - \sigma] \ln H + \left\| \sum_{j=1}^N \sigma_{ij} \ln \left(\sigma_{ij} \frac{m_i}{m_j} \right) \right\| \right\| \right) \quad (28)$$

Note that this gives sectoral gross output as a function of only parameters. Next, I use this expression into the definition of aggregate GDP (in logs) to obtain an equilibrium aggregate output:

$$\begin{aligned} Y &= \mathcal{L} c_1^{\beta_1} c_2^{\beta_2} \dots c_N^{\beta_N} \\ y = \ln Y &= \ln \mathcal{L} + \sum_{i=1}^N \beta_i \ln c_i \\ y &= \ln \mathcal{L} + \sum_{i=1}^N \beta_i \ln \left(\frac{\beta_i Q_i}{m_i} \right) \\ y &= \ln \mathcal{L} + \sum_{i=1}^N \beta_i \ln \left(\frac{\beta_i}{m_i} \right) + \beta' \mathbf{q} \end{aligned}$$

Substituting 28 into the above expression for GDP, we have:

$$\begin{aligned}
y &= \ln \mathcal{L} + \sum_{i=1}^N \beta_i \ln \left(\frac{\beta_i}{m_i} \right) + \beta' (\mathbf{I} - \mathbf{B})^{-1} \left(\|(1 - \sigma_i) a_i\| + \dots \right. \\
&\quad \left. \|(1 - \sigma_i) \ln v_i\| + [\mathbf{1} - \sigma] \ln H + \left\| \sum_{j=1}^N \sigma_{ij} \ln \left(\sigma_{ij} \frac{m_i}{m_j} \right) \right\| \right) \\
y &= \ln \mathcal{L} + \sum_{i=1}^N \beta_i \ln \left(\frac{\beta_i}{m_i} \right) + \mathbf{m} \left(\|(1 - \sigma_i) a_i\| + \|(1 - \sigma_i) \ln v_i\| + \dots \right. \\
&\quad \left. [\mathbf{1} - \sigma] \ln H + \left\| \sum_{j=1}^N \sigma_{ij} \ln \left(\sigma_{ij} \frac{m_i}{m_j} \right) \right\| \right) \\
y &= \ln \bar{\mathcal{A}} + \sum_{i=1}^N v_i \ln A_i + \sum_{i=1}^N v_i \ln v_i + \sum_{i=1}^N v_i \ln H \\
y &= \ln \bar{\mathcal{A}} + \sum_{i=1}^N v_i (\ln A_i + \ln v_i + \ln H) \\
y &= \ln \bar{\mathcal{A}} + \sum_{i=1}^N v_i \ln (A_i v_i H)
\end{aligned}$$

Thus, taking the exponential function in both sides, I arrive to:

$$Y = \mathcal{A} H = \bar{\mathcal{A}} (A_1 v_1)^{v_1} (A_2 v_2)^{v_2} \dots (A_N v_N)^{v_N} H,$$

or in logs:

$$\ln Y = \ln \bar{\mathcal{A}} + \mathbf{v} \mathbf{a} + \mathbf{v} \ln \mathbf{v}' + \ln H.$$

where: $\ln \bar{\mathcal{A}} = \ln \mathcal{L} + \sum_{i=1}^N \beta_i \ln \left(\frac{\beta_i}{m_i} \right) + \sum_{i=1}^N m_i \sum_{j=1}^N \sigma_{ij} \ln \left(\sigma_{ij} \frac{m_i}{m_j} \right)$.

We can always choose the value of \mathcal{L} to make $\ln \bar{\mathcal{A}} = 0$. By recognizing that the elements of the vector of influence represent the value added shares, we have proven that the GDP in the input-output economy is the same as in the value added economy.

Consumption is a constant fraction of gross output. To see this, note that the Dommar weights are given by $m_i = p_i Q_i / Y$, and that the FOC from the composite producer is $\beta_i = p_i c_i / Y$. Combining the two, we have: $Q_i / c_i = m_i / \beta_i$.